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Fuzzy Nature of a Conjecture on Nonelementary Integrals

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Abstract
Fuzziness is also seen in the nature of the integrals of some special type of elementary functions in binary nature i.e. it doesn't lie between yes and no but it is either yes or no. In this case the fuzziness behaves like a member of a classical set or crisp set and as per mathematical logic it is either true (elementary) or false (nonelementary) i.e. either belongs to the set or doesn't belong to the set, where the set denotes the collection of all elementary functions. In this paper we have propounded a conjecture on antiderivative containing the functions made of inverse trigonometric functions and polynomials, where a particular function written in the numerator is made up by a composition of two functions, the inverse trigonometric functions and a polynomial function. The integrand contains two polynomials, which may or may not be equal, one as an argument in the inverse trigonometric function and another as a denominator of the integrand. The computing software Mathematica has played a vital role in integrating the complex functions originating as a particular case of the proposed conjecture. It has been found that the nature of the functions written as integrands varies as we increase the degree of the polynomials written both in numerator and denominator. Two very interesting integrals have been found in the study, which are always elementary containing inverse tangent and cotangent functions. The paper ends with the conclusion, its limitations and the scope of further research.

Keywords: Conjecture, elementary and nonelementary functions, polylogarithm function, dilogarithm function, elliptic function, hypergeometric function.

1. Introduction

Fuzziness is not only a study of numbers lying between zero and one but also a study on the nature of some mathematical operations and their results for some properties of functions originating from the antiderivatives of special functions made of following some particular patterns (Fuzzy logic-Wikipedia contributors, 2024). The study of elementary and nonelementary functions (integrals) is such topic in mathematics, where we can apply the concept of such fuzziness in binary nature.

The study of elementary and nonelementary functions originating from antiderivatives has been a subject of many efforts in the past. The first example leading beyond the region of elementary functions is the elliptic integrals. In general they cannot be expressed in terms of elementary functions. The first reported study of such integrals was due to John Wallis in 1655, when he started to study to find the arc length of an ellipse (Elementary function-Wikipedia contributors, 2024; Marchisotto et al., 1994; Nonelementary integrals-Wikipedia contributors, 2024; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972). Yadav & Sen (2008) has propounded such six nonelementary integrals as six conjectures.

In mathematics a conjecture means to guess about something without a real proof. It, in general, states a mathematical statement that is proposed on a tentative basis without a solid proof (Conjecture-Wikipedia contributors, 2024). The study of nature of such antiderivative of some special type of functions as conjectures has attracted the researchers in recent decades in context of elementary and nonelementary functions. In the paper we have also stated such functions and their integrals, as conjecture, because we have not discussed the cases for general polynomials of arbitrary degree, which is almost not possible.

2. Preliminary Ideas

For the present study, we need to know some basic concepts of the elementary and nonelementary functions and about some special functions. Elementary and nonelementary functions were introduced by Joseph Liouville in a series of papers from 1833 to 1841 and the algebraic treatment of elementary functions was started by Joseph Fels Ritt in 1930s (Hardy, 2018; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav & Sen, 2012; Yadav, 2023). They are stated and defined as:

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2.1 Elementary Function: An elementary function is a single variable real or complex valued function, which can be expressed as the sums, differences, products, divisions, roots and composition of finitely many polynomials, rational, trigonometric, hyperbolic, exponential, and their inverse functions. For example, x^2+3, $\sqrt{x^2+x+1}$, e^x, $\log(x+1)$, $sinx+2x^2+x$, $\int x\, sinx\, dx$, $\int 3x\, e^{x^2}\, dx$, π, e, 4, $sinh3x$, $arcsin5x$, $|x|$, etc. are elementary functions. But every function need not be an elementary function. For example, the integral $\int e^{-t^2}dt$ is not an elementary function (Elementary function-Wikipedia contributors, 2024; Hardy, 2018; Kasper, 1980; Lutfi, 2016; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav & Sen, 2012; Yadav, 2023).

2.2 Nonelementary Function: If a function is expressed in closed form, its derivative can be expressed in closed form. But its integral may or may not be expressed in closed form. One such example of an elementary function, whose antiderivative does not have a closed form expression is e^{-x^2}, whose one antiderivative is the error function

$$\mathrm{erf}(x)=\frac{1}{\pi}\int_0^x e^{-t^2}dt.$$

Such integrals are known as nonelementary integrals or nonelementary functions (Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav & Sen, 2012; Yadav, 2023). So for the present study, a nonelementary function of a given elementary function is an antiderivative that is not an elementary function and that cannot be expressed in closed form expression (Hardy, 2018; Marchisotto et al., 1994; Yadav et al., 2012; Yadav, 2023).

3. Methodology

In general when we apply the Mathematica code to integrate a function f(x) with respect to x using '*In[i]:Integrate[f[x], x]*', we get its integral in terms of elementary functions. But when the result comes in terms of some special form of functions like hypergeometric, dilogarithm, polylogarithm, elliptic, etc., it means the integral is not elementary function. This concept will be used in the present paper. Although many special functions are elementary functions for particular values but when we use Mathematica, it comes either as an elementary function or as a special function. If it is any special function, it means the integral is nonelementary (Elliptic function-Wikipedia contributors, 2024; Hypergeometric function -Wikipedia contributors, 2024; Nonelementary integrals-Wikipedia contributors, 2024; Polylogarithm function-Wikipedia contributors, 2024).

4. Discussion

Previously Yadav & Sen (2008, 2012) propounded six conjectures on nonelementary functions and proved them using strong Liouville's theorem and its particular properties. Then Chaudhary & Yadav (2024), Yadav & Chaudhary (2024), and Yadav & Yadav (2024a, 2024b) proposed two more conjectures based on those functions, which was left earlier by Yadav & Sen (2008, 2012) and proved them using strong Liouville's theorem, Laplace's theorem and mathematica software. In continuation, we are here studying the nature of a special type of integrand in an antiderivative.

Before propounding a conjecture, let us consider the antiderivatives of the following indefinite integral

$$\int \frac{fog(x)}{h(x)}dx \qquad (1)$$

where $fog(x)=f[g(x)]$ is a composition of two functions: inverse trigonometric function f(x) and a polynomial g(x), where both g(x) and h(x) are polynomial of degree greater than or equal to one. The two polynomials g(x) and h(x) are arbitrary and are independent of each other. They may or may not be related to each other.

To evaluate the above integration, first suppose that the integral is denoted by I. We know that there are six types of inverse trigonometric functions, so there will be six cases for different inverse trigonometric functions. Let us consider them one by one.

Case-I: When we take $f(x)=sin^{-1}x$. The integral (1) becomes

$$\mathrm{I}=\int \frac{\sin^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}}\mathrm{dx} \qquad (1.1)$$

Let us consider a special case when g(x) = h(x) = x. Then from (1.1), we have

$$\mathrm{I}=\int \frac{\sin^{-1}\mathrm{x}}{\mathrm{x}}\mathrm{dx}$$

Putting $sin^{-1}x=z$, we get

$$\mathrm{I}=\int \frac{\mathrm{z.\cos z}}{\mathrm{\sin z}}\mathrm{dz}$$

Using Mathematica, we get

$$\mathrm{In[1]: Integrate}[\frac{z*\mathrm{Cos}[z]}{\mathrm{Sin}[z]},z]$$

$$\mathrm{Out}[1]\colon z\mathrm{Log}[1-e^{2iz}]-\frac{1}{2}i(z^2+\mathrm{PolyLog}[2,e^{2iz}])$$

where $\mathrm{PolyLog}[2,e^{2iz}]$ is a polylogarithm function $\mathrm{Li}_2(e^{2iz})$. We know that the polylogarithm function $\mathrm{Li}_n(z)$ is elementary for only special values of n. The special case n = 1 involves the ordinary natural logarithm and the special cases for n = 2 and n = 3 are called dilogarithm (Polylogarithm - Wikipedia contributors, 2024). Thus the above integral is obviously not an elementary function. We can find the integral of $\frac{\sin^{-1}x}{x}$ directly of the integrand using mathematica code and will get the same result.

Let us consider another special case, when h(x) is the derivative of g(x) like g(x) = x and h(x) = 1, although it is beyond our assumption as we have taken both g and h of degree at least equal to one. Then from (1.1), we have

$$\mathrm{I}=\int\frac{\sin^{-1}[\mathrm{x}]}{1}\mathrm{dx}=\int\sin^{-1}[\mathrm{x}]\mathrm{dx}$$

Integrating using Mathematica, we get

$$\mathrm{In}[2]\colon \mathrm{Integrate}[\mathrm{ArcSin}[x],x]$$

$$\mathrm{Out}[2]\colon \sqrt{1-x^2}+x\mathrm{ArcSin}[x]$$

which is elementary. This is the reason that we have taken the degree of polynomials greater than or equal to one. Let us take another case when g(x) = x^2 and h(x) = 2x, here h(x) is the first derivative of g(x). Then from (1.1), we get

$$\mathrm{I}=\int\frac{\sin^{-1}[\mathrm{x}^2]}{2\mathrm{x}}\mathrm{dx}$$

Integrating it using Mathematica, we get

$$\mathrm{In}[3]\colon \mathrm{Integrate}[\frac{\mathrm{ArcSin}[x^2]}{2*x},x]$$

$$\mathrm{Out}[3]\colon \frac{1}{4}(\mathrm{ArcSin}[x^2]\mathrm{Log}[1-e^{2i\mathrm{ArcSin}[x^2]}]$$

$$-\frac{1}{2}i(\mathrm{ArcSin}[x^2]^2+\mathrm{PolyLog}[2,e^{2i\mathrm{ArcSin}[x^2]}]))$$

obviously it is nonelementary, as discussed earlier.

Now let us consider some arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2$+1. Then from (1.1), we get

$$\mathrm{I}=\int\frac{\sin^{-1}[\mathrm{x}^2+2\mathrm{x}]}{2\mathrm{x}^2+1}\mathrm{dx}$$

For such polynomial the Mathematica doesn't produce any output but for another polynomial in mixed term, it is sometimes elementary and sometimes nonelementary integrals. So we cannot generalize it for mixed terms polynomials. So let us simplify it for other polynomial having only one term with highest degree as

$$\mathrm{I}=\int\frac{\sin^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\mathrm{dx}$$

Using Mathematica, we get

$$\mathrm{In}[4]\colon \mathrm{Integrate}[\frac{\mathrm{ArcSin}[x^3]}{x^2},x]$$

$$\mathrm{Out}[4]\colon -\frac{\mathrm{ArcSin}[x^3]}{x}$$
$$+\frac{1}{\sqrt{1-x^6}}i3^{3/4}\sqrt{(-1)^{5/6}(-1+x^2)}\sqrt{1+x^2+x^4}$$

$$\mathrm{EllipticF}[\mathrm{ArcSin}[\frac{\sqrt{-(-1)^{5/6}-ix^2}}{3^{1/4}}],(-1)^{1/3}]$$

which is in Elliptic functions form, which is itself nonelementary. Elliptic functions are special cases of meromorphic functions and they have been derived from elliptic integrals (Elliptic function - Wikipedia contributors, 2024), which are themselves nonelementary. Considering some more antiderivatives with higher degree polynomials and their integrals, we find that it is nonelementary for all m and n greater than or equal to 1. Thus we can state that for f(x) = $\sin^{-1}x$ and for any polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is always nonelementary provided that the polynomials have only one leading term with the highest degree of it.

Case-II: When we take $f(x)=cos^{-1}x$. The integral (1) becomes

$$\mathrm{I}=\int\frac{\cos^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}}\mathrm{dx}\qquad(1.2)$$

Let us consider a case when both g(x) = h(x) = x. Then from (1.2), we get

$$\mathrm{I}=\int\frac{cos^{-1}x}{x}dx$$

Putting $cos^{-1}x=z$, we get

$$\mathrm{I}=-\int\frac{z.sinz}{cosz}dz$$

Integrating using Mathematica, we get

$$\mathrm{In}[5]\text{: Integrate}[-1 * \frac{z * \mathrm{Sin}[z]}{\mathrm{Cos}[z]}, z]$$

$$\mathrm{Out}[5]\text{: } -\frac{iz^2}{2} + z\mathrm{Log}[1 + e^{2iz}] - \frac{1}{2} i\mathrm{PolyLog}[2, -e^{2iz}]$$

where $\mathrm{PolyLog}[2, -e^{2iz}]$ is a polylogarithm function $\mathrm{Li}_2(e^{2iz})$. Thus the above integral is not an elementary function.

Taking another special case like g(x) = x and h(x) = 1. Then from (1.2), we have

$$\mathrm{I} = \int \frac{\cos^{-1}[\mathrm{x}]}{1} \mathrm{dx} = \int \cos^{-1}[\mathrm{x}] \, \mathrm{dx}$$

Integrating using Mathematica, we get

$$\mathrm{In}[6]\text{: Integrate}[\mathrm{ArcCos}[x], x]$$

$$\mathrm{Out}[6]\text{: } -\sqrt{1 - x^2} + x\mathrm{ArcCos}[x]$$

which is elementary but the case beyond our assumptions. Let us take another case when g(x) = x^2 and h(x) = 2x. Then from (1.2), we get

$$\mathrm{I} = \int \frac{\cos^{-1}[\mathrm{x}^2]}{2\mathrm{x}} \mathrm{dx}$$

Integrating it using Mathematica, we get

$$\mathrm{In}[7]\text{: Integrate}[\frac{\mathrm{ArcCos}[x^2]}{2 * x}, x]$$

$$\mathrm{Out}[7]\text{: } -\frac{1}{8} i(\mathrm{ArcCos}[x^2](\mathrm{ArcCos}[x^2] + 2i\mathrm{Log}[1 + e^{2i\mathrm{ArcCos}[x^2]}]) + \mathrm{PolyLog}[2, -e^{2i\mathrm{ArcCos}[x^2]}])$$

obviously it is nonelementary, as discussed earlier.

Let us consider arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2$+1. Then from (1.2), we get

$$\mathrm{I} = \int \frac{\cos^{-1}[\mathrm{x}^2 + 2\mathrm{x}]}{2\mathrm{x}^2 + 1} \mathrm{dx}$$

In this case the Mathematica doesn't produce any output. But for some polynomials it gives output in either in elementary or in nonelementary form, which cannot be generalize. So let us simplify it for simple polynomial as

$$\mathrm{I} = \int \frac{\cos^{-1}[\mathrm{x}^3]}{\mathrm{x}^2} \mathrm{dx}$$

Using Mathematica code, we get

$$\mathrm{In}[8]\text{: Integrate}[\frac{\mathrm{ArcCos}[x^3]}{x^2}, x]$$

$$\mathrm{Out}[8]\text{: } -\frac{\mathrm{ArcCos}[x^3]}{x} - \frac{1}{\sqrt{1 - x^6}} i3^{3/4} \sqrt{(-1)^{5/6}(-1 + x^2)}\sqrt{1 + x^2 + x^4}$$

$$\mathrm{EllipticF}[\mathrm{ArcSin}[\frac{\sqrt{-(-1)^{5/6} - ix^2}}{3^{1/4}}], (-1)^{1/3}]$$

which is in Elliptic functions form and is itself nonelementary, as has been discussed earlier in case-I. Considering more antiderivatives as above with higher degree polynomials and their integrals, we find that it is nonelementary for all m and n greater than or equal to 1.

Thus we can state that for f(x) = $\cos^{-1}$x and for any polynomial g(x), h(x) of degree greater than or equal to one, the integral (1) is always nonelementary, provided that the polynomials have only one leading coefficient term with the highest degree.

Case-III: When we take $f(x) = tan^{-1}x$, the integral (1) becomes

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}} \mathrm{dx} \qquad (1.3)$$

As previously considered a special case for both g(x) = h(x) = x, we get from (1.3)

$$I = \int \frac{tan^{-1}x}{x} dx$$

Applying Mathematica, we get

$$\mathrm{In}[9]\text{: Integrate}[\frac{\mathrm{ArcTan}[x]}{x}, x]$$

$$\mathrm{Out}[9]\text{: } \frac{1}{2} i(\mathrm{PolyLog}[2, -ix] - \mathrm{PolyLog}[2, ix])$$

which is nonelementary. Let us verify it after substitution. Putting $tan^{-1}x = z$, we get

$$I = \int \frac{z.sec^2z}{tanz} dz$$

Integrating using Mathematica, we get

$$\mathrm{In}[10]\text{: Integrate}[\frac{z * \mathrm{Sec}[z] * \mathrm{Sec}[z]}{\mathrm{Tan}[z]}, z]$$

$$\mathrm{Out}[10]\text{: } \frac{1}{2}(2z(\mathrm{Log}[1 - e^{2iz}] - \mathrm{Log}[1 + e^{2iz}]) + i(\mathrm{PolyLog}[2, -e^{2iz}] - \mathrm{PolyLog}[2, e^{2iz}]))$$

where $\mathrm{PolyLog}[2, e^{2iz}]$ and $\mathrm{PolyLog}[2, -e^{2iz}]$ are polylogarithm function $\mathrm{Li}_2(\mathrm{e}^{2\mathrm{iz}})$, which are not elementary functions i.e., substitution doesn't change the nature of the integrals, which is a universal property of integration.

If we consider another case like g(x) = x and h(x) = 1, beyond our assumption. Then from (1.3), we have

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{x}]}{1} \mathrm{dx} = \int \tan^{-1}[\mathrm{x}] \mathrm{dx}$$

Integrating using Mathematica, it we get

$$\mathrm{In}[11]\colon \mathrm{Integrate}[\mathrm{ArcTan}[x], x]$$

$$\mathrm{Out}[11]\colon x\mathrm{ArcTan}[x] - \frac{1}{2}\mathrm{Log}[1 + x^2]$$

which is elementary. Let us take another case when g(x) = x^2 and h(x) = 2x, then from (1.3), we get

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{x}^2]}{2\mathrm{x}} \mathrm{dx}$$

Using Mathematica for integration, we get

$$\mathrm{In}[12]\colon \mathrm{Integrate}[\frac{\mathrm{ArcTan}[x^2]}{2 * x}, x]$$

$$\mathrm{Out}[12]\colon \frac{1}{8} i(\mathrm{PolyLog}[2, -ix^2] - \mathrm{PolyLog}[2, ix^2])$$

obviously it is nonelementary, as discussed earlier. Let us take one more example

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{x}^2]}{\mathrm{x}^2} \mathrm{dx}$$

Using Mathematica, we get

$$\mathrm{In}[13]\colon \mathrm{Integrate}[\frac{\mathrm{ArcTan}[x^2]}{x^2}, x]$$

$$\mathrm{Out}[13]\colon \frac{1}{4x}(-4\mathrm{ArcTan}[x^2] + \sqrt{2}x(-2\mathrm{ArcTan}[1 - \sqrt{2}x] + 2\mathrm{ArcTan}[1 + \sqrt{2}x] - \mathrm{Log}[1 - \sqrt{2}x + x^2] + \mathrm{Log}[1 + \sqrt{2}x + x^2]))$$

which is elementary. Let us take some arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2+1$. Then from (1.3), we get

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{x}^2 + 2\mathrm{x}]}{2\mathrm{x}^2 + 1} \mathrm{dx}$$

In this case the Mathematica doesn't produce any output although for some integrands, it gives output in either elementary or in nonelementary form. So let us consider another integrand as

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{x}^3]}{\mathrm{x}^2} \mathrm{dx}$$

Using Mathematica, we get

$$\mathrm{In}[14]\colon \mathrm{Integrate}[\frac{\mathrm{ArcTan}[x^3]}{x^2}, x]$$

$$\mathrm{Out}[14]\colon -\frac{1}{4x}(2\sqrt{3}x\mathrm{ArcTan}[\sqrt{3} - 2x] + 4\mathrm{ArcTan}[x^3] + x(2\sqrt{3}\mathrm{ArcTan}[\sqrt{3} + 2x] - 2\mathrm{Log}[1 + x^2] + \mathrm{Log}[1 - \sqrt{3}x + x^2] + \mathrm{Log}[1 + \sqrt{3}x + x^2]))$$

which is elementary. Let us verify it for one more example as

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{x}^3]}{\mathrm{x}^3} \mathrm{dx}$$

We get from Mathematica that

$$\mathrm{In}[15]\colon \mathrm{Integrate}[\frac{\mathrm{ArcTan}[x^3]}{x^3}, x]$$

$$\mathrm{Out}[15]\colon \frac{1}{8}(-2\mathrm{ArcTan}[\sqrt{3} - 2x] + 4\mathrm{ArcTan}[x] - \frac{4\mathrm{ArcTan}[x^3]}{x^2} + 2\mathrm{ArcTan}[\sqrt{3} + 2x] - \sqrt{3}\mathrm{Log}[1 - \sqrt{3}x + x^2] + \sqrt{3}\mathrm{Log}[1 + \sqrt{3}x + x^2])$$

which is elementary. Again one more example

$$\mathrm{I} = \int \frac{\tan^{-1}[\mathrm{x}^4]}{\mathrm{x}^4} \mathrm{dx}$$

We get using Mathematica code,

$$\mathrm{In}[16]\colon \mathrm{Integrate}[\frac{\mathrm{ArcTan}[x^4]}{x^4}, x]$$

$$Out[16]\colon -\frac{\mathrm{ArcTan}[x^4]}{3x^3} + \frac{1}{3}\mathrm{ArcTan}[\mathrm{Sec}[\frac{\pi}{8}](x - \mathrm{Sin}[\frac{\pi}{8}])]\mathrm{Cos}[\frac{\pi}{8}] + \frac{1}{3}\mathrm{ArcTan}[\mathrm{Sec}[\frac{\pi}{8}](x + \mathrm{Sin}[\frac{\pi}{8}])]\mathrm{Cos}[\frac{\pi}{8}]$$

$$-\frac{1}{6}\mathrm{Cos}\left[\frac{\pi}{8}\right]\mathrm{Log}\left[1 + x^2 - 2x\mathrm{Cos}\left[\frac{\pi}{8}\right]\right] + \frac{1}{6}\mathrm{Cos}\left[\frac{\pi}{8}\right]\mathrm{Log}\left[1 + x^2 + 2x\mathrm{Cos}\left[\frac{\pi}{8}\right]\right] +$$

$$\frac{1}{3}\mathrm{ArcTan}\left[\left(x-\mathrm{Cos}\left[\frac{\pi}{8}\right]\right)\mathrm{Csc}\left[\frac{\pi}{8}\right]\right]\mathrm{Sin}\left[\frac{\pi}{8}\right]$$

$$+\frac{1}{3}\mathrm{ArcTan}\left[\left(x+\mathrm{Cos}\left[\frac{\pi}{8}\right]\right)\mathrm{Csc}\left[\frac{\pi}{8}\right]\right]\mathrm{Sin}\left[\frac{\pi}{8}\right]-$$

$$\frac{1}{6}\mathrm{Log}\left[1+x^2-2x\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]\mathrm{Sin}\left[\frac{\pi}{8}\right]$$

$$+\frac{1}{6}\mathrm{Log}[1+x^2+2x\mathrm{Sin}[\frac{\pi}{8}]]\mathrm{Sin}[\frac{\pi}{8}]$$

which is elementary. Let us take one more example

$$\mathrm{I}=\int\frac{\tan^{-1}[\mathrm{x}^4]}{\mathrm{x}^3}\mathrm{dx}$$

Using Mathematica, we get

$$\mathrm{In}[17]:\mathrm{Integrate}\left[\frac{\mathrm{ArcTan}[x^4]}{x^3},x\right]$$

$$Out[17]:-\frac{1}{8x^2}(4\mathrm{ArcTan}[x^4]+\sqrt{2}x^2(2\mathrm{ArcTan}[\mathrm{Cot}[\frac{\pi}{8}]-x\mathrm{Csc}[\frac{\pi}{8}]]+2\mathrm{ArcTan}[\mathrm{Cot}[\frac{\pi}{8}]+x\mathrm{Csc}[\frac{\pi}{8}]]$$
$$-2\mathrm{ArcTan}[x\mathrm{Sec}[\frac{\pi}{8}]-\mathrm{Tan}[\frac{\pi}{8}]]+2\mathrm{ArcTan}[x\mathrm{Sec}[\frac{\pi}{8}]+\mathrm{Tan}[\frac{\pi}{8}]]+\mathrm{Log}[1+x^2-2x\mathrm{Cos}[\frac{\pi}{8}]]+\mathrm{Log}[1+x^2$$
$$+2x\mathrm{Cos}[\frac{\pi}{8}]]-\mathrm{Log}[1+x^2-2x\mathrm{Sin}[\frac{\pi}{8}]]-\mathrm{Log}[1+x^2+2x\mathrm{Sin}[\frac{\pi}{8}]]))$$

which is elementary.

Thus we can state that for f(x) = $\tan^{-1}$x and for polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is elementary for some cases and nonelementary for some other cases. It cannot be unified in a single statement. In all cases the polynomials should have only one leading term with the highest degree. This is the fuzziness of the integral discussed in (1).

But a particular chain is seen in the integral of the form

$$\mathrm{I}=\int\frac{\tan^{-1}[\mathrm{x}^\mathrm{n}]}{\mathrm{x}^\mathrm{m}}\mathrm{dx}$$

that it is elementary for all values of n and m except for n = 1, m = 1 and n = 2, m = 1, for which it is nonelementary under our assumptions in (1).

Case-IV: When we take $f(x)=cot^{-1}x$, the integral (1) becomes

$$\mathrm{I}=\int\frac{\cot^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}}\mathrm{dx}\qquad(1.4)$$

For special case for both g(x) = h(x) = x, we get from (1.4)

$$I=\int\frac{cot^{-1}x}{x}dx$$

Applying Mathematica, we get

$$\mathrm{In}[18]:\mathrm{Integrate}[\frac{\mathrm{ArcCot}[x]}{x},x]$$

$$\mathrm{Out}[18]:-\frac{1}{2}i(\mathrm{PolyLog}[2,-\frac{i}{x}]-\mathrm{PolyLog}[2,\frac{i}{x}])$$

where $\mathrm{PolyLog}\left[2,\frac{i}{x}\right]$ and $\mathrm{PolyLog}\left[2,-\frac{i}{x}\right]$ are polylogarithm function, which are not elementary functions. So the above integral is nonelementary.

If we consider another case like g(x) = x and h(x) = 1, although it is beyond our assumption as we have taken both g and h of degree at least equal to one. Then from (1.4), we have

$$\mathrm{I}=\int\frac{\cot^{-1}[\mathrm{x}]}{1}\mathrm{dx}=\int\cot^{-1}[\mathrm{x}]\mathrm{dx}$$

Integrating using Mathematica, we get

$$\mathrm{In}[19]:\mathrm{Integrate}[\mathrm{ArcCot}[x],x]$$

$$\mathrm{Out}[19]:x\mathrm{ArcCot}[x]+\frac{1}{2}\mathrm{Log}[1+x^2]$$

which is elementary. Let us take g(x) = x^2 and h(x) = 2x. Then from (1.4), we get

$$\mathrm{I}=\int\frac{\cot^{-1}[\mathrm{x}^2]}{2\mathrm{x}}\mathrm{dx}$$

Using Mathematica for integration, we get

$$\mathrm{In}[20]:\mathrm{Integrate}[\frac{\mathrm{ArcCot}[x^2]}{2*x},x]$$

$$\mathrm{Out}[20]:\frac{1}{8}i(-\mathrm{PolyLog}[2,-\frac{i}{x^2}]+\mathrm{PolyLog}[2,\frac{i}{x^2}])$$

obviously it is nonelementary, as discussed earlier. Let us take another example

$$\mathrm{I}=\int\frac{\cot^{-1}[\mathrm{x}^2]}{\mathrm{x}^2}\mathrm{dx}$$

and using Mathematica, we get

$$\mathrm{In}[21]:\mathrm{Integrate}[\frac{\mathrm{ArcCot}[x^2]}{x^2},x]$$

$$\text{Out}[21]\text{:}\frac{1}{4x}\Big(-4\text{ArcCot}[x^2] + \sqrt{2}x\big(2\text{ArcTan}\big[1-\sqrt{2}x\big] - 2\text{ArcTan}\big[1+\sqrt{2}x\big] + \text{Log}\big[1-\sqrt{2}x+x^2\big] - \text{Log}\big[1+\sqrt{2}x+x^2\big]\big)\Big)$$

which is elementary. Let us take some arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2+1$. Then from (1.4), we get

$$\mathrm{I} = \int \frac{\cot^{-1}[\mathrm{x}^2+2\mathrm{x}]}{2\mathrm{x}^2+1}\,\mathrm{dx}$$

In this case the Mathematica doesn't produce any output although for some integrands, it gives output in either elementary or in nonelementary form. So let us simplify it for

$$\mathrm{I} = \int \frac{\cot^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\,\mathrm{dx}$$

Using Mathematica code, we get

$$\text{In}[22]\text{: Integrate}[\frac{\text{ArcCot}[x^3]}{x^2}, x]$$

$$\text{Out}[22]\text{:}\frac{1}{4x}(-4\text{ArcCot}[x^3] + x(2\sqrt{3}\text{ArcTan}[\sqrt{3} - 2x] + 2\sqrt{3}\text{ArcTan}[\sqrt{3}+2x] - 2\text{Log}[1+x^2] + \text{Log}[1-\sqrt{3}x + x^2] + \text{Log}[1+\sqrt{3}x+x^2]))$$

which is elementary. Let us verify it for one more example

$$\mathrm{I} = \int \frac{\cot^{-1}[\mathrm{x}^3]}{\mathrm{x}^3}\,\mathrm{dx}$$

We get using Mathematica that

$$\text{In}[23]\text{: Integrate}[\frac{\text{ArcCot}[x^3]}{x^3}, x]$$

$$\text{Out}[23]\text{:}\frac{1}{8x^2}(-4\text{ArcCot}[x^3] + x^2(2\text{ArcTan}[\sqrt{3} - 2x] - 4\text{ArcTan}[x] - 2\text{ArcTan}[\sqrt{3} + 2x] + \sqrt{3}\text{Log}[1-\sqrt{3}x+x^2] - \sqrt{3}\text{Log}[1+\sqrt{3}x+x^2]))$$

which is elementary. Again taking one more example

$$\mathrm{I} = \int \frac{\cot^{-1}[\mathrm{x}^4]}{\mathrm{x}^4}\,\mathrm{dx}$$

We get using Mathematica code that

$$\text{In}[24]\text{: Integrate}[\frac{\text{ArcCot}[x^4]}{x^4}, x]$$

$$Out[24]\text{:} -\frac{\text{ArcCot}[x^4]}{3x^3} - \frac{1}{3}\text{ArcTan}[\text{Sec}[\frac{\pi}{8}](x - \text{Sin}[\frac{\pi}{8}])]\text{Cos}[\frac{\pi}{8}] - \frac{1}{3}\text{ArcTan}[\text{Sec}[\frac{\pi}{8}](x + \text{Sin}[\frac{\pi}{8}])]\text{Cos}[\frac{\pi}{8}] + \frac{1}{6}\text{Cos}[\frac{\pi}{8}]\text{Log}[1 + x^2 - 2x\text{Cos}[\frac{\pi}{8}]] - \frac{1}{6}\text{Cos}[\frac{\pi}{8}]\text{Log}[1 + x^2 + 2x\text{Cos}[\frac{\pi}{8}]] - \frac{1}{3}\text{ArcTan}[(x - \text{Cos}[\frac{\pi}{8}])\text{Csc}[\frac{\pi}{8}]]\text{Sin}[\frac{\pi}{8}] - \frac{1}{3}\text{ArcTan}[(x + \text{Cos}[\frac{\pi}{8}])\text{Csc}[\frac{\pi}{8}]]\text{Sin}[\frac{\pi}{8}] + \frac{1}{6}\text{Log}[1 + x^2 - 2x\text{Sin}[\frac{\pi}{8}]]\text{Sin}[\frac{\pi}{8}] - \frac{1}{6}\text{Log}[1 + x^2 + 2x\text{Sin}[\frac{\pi}{8}]]\text{Sin}[\frac{\pi}{8}]$$

which is elementary. Let us take one more example

$$\mathrm{I} = \int \frac{\cot^{-1}[\mathrm{x}^4]}{\mathrm{x}^3}\,\mathrm{dx}$$

and using Mathematica, we get

$$\text{In}[25]\text{: Integrate}\left[\frac{\text{ArcCot}[x^4]}{x^3}, x\right]$$

$$Out[25]\text{:}\frac{1}{8x^2}(-4\text{ArcCot}[x^4] + \sqrt{2}x^2(2\text{ArcTan}[\text{Cot}[\frac{\pi}{8}] - x\text{Csc}[\frac{\pi}{8}]] + 2\text{ArcTan}[\text{Cot}[\frac{\pi}{8}] + x\text{Csc}[\frac{\pi}{8}]] - 2\text{ArcTan}[x\text{Sec}[\frac{\pi}{8}] - \text{Tan}[\frac{\pi}{8}]] + 2\text{ArcTan}[x\text{Sec}[\frac{\pi}{8}] + \text{Tan}[\frac{\pi}{8}]] + \text{Log}[1 + x^2 - 2x\text{Cos}[\frac{\pi}{8}]] + \text{Log}[1 + x^2 + 2x\text{Cos}[\frac{\pi}{8}]] - \text{Log}[1 + x^2 - 2x\text{Sin}[\frac{\pi}{8}]] - \text{Log}[1 + x^2 + 2x\text{Sin}[\frac{\pi}{8}]]))$$

which is elementary.

Thus we can state that for f(x) = $\cot^{-1}x$ and for polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is elementary for some cases and nonelementary for some other cases. In all cases the polynomials have only the leading coefficient term with highest degree. It cannot be unified in a single statement. But a particular chain is again seen in the integral of the form

$$\mathrm{I} = \int \frac{\cot^{-1}[x^n]}{x^m} dx$$

that it is elementary for all values of n and m except for n = 1, m = 1 and n = 2, m = 1, for which it is nonelementary under our assumptions.

Case-V: When we take $f(x) = cosec^{-1}x$, the integral (1) becomes

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[g(x)]}{h(x)} dx \qquad (1.5)$$

For special case for both g(x) = h(x) = x, we get from (1.5)

$$I = \int \frac{cosec^{-1}x}{x} dx$$

Applying Mathematica, we get

$$\mathrm{In}[26]{:}\ \mathrm{Integrate}[\frac{\mathrm{ArcCsc}[x]}{x}, x]$$

$$\mathrm{Out}[26]{:}\ -\mathrm{ArcCsc}[x]\mathrm{Log}[1 - e^{2i\mathrm{ArcCsc}[x]}] + \frac{1}{2}i(\mathrm{ArcCsc}[x]^2 + \mathrm{PolyLog}[2, e^{2i\mathrm{ArcCsc}[x]}])$$

which is in the form of polylogarithm and thus nonelementary.

If we consider another case like g(x) = x and h(x) = 1, although it is beyond our assumption as we have taken both g and h of degree at least equal to one. Then from (1.5), we have

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[x]}{1} dx = \int \mathrm{cosec}^{-1}[x] dx$$

Integrating using Mathematica, it we get

$$\mathrm{In}[27]{:}\ \mathrm{Integrate}[\mathrm{ArcCsc}[x], x]$$

$$\mathrm{Out}[27]{:}\ x\mathrm{ArcCsc}[x] + \frac{\sqrt{-1+x^2}\mathrm{Log}[2(x+\sqrt{-1+x^2})]}{\sqrt{1-\frac{1}{x^2}}x}$$

which is elementary. Taking another case when g(x) = x^2 and h(x) = 2x. Then from (1.5), we get

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[x^2]}{2x} dx$$

Using Mathematica for integration, we get

$$\mathrm{In}[28]{:}\ \mathrm{Integrate}[\frac{\mathrm{ArcCsc}[x^2]}{2*x}, x]$$

$$\mathrm{Out}[28]{:}\ \frac{1}{8}i(\mathrm{ArcCsc}[x^2](\mathrm{ArcCsc}[x^2] + 2i\mathrm{Log}[1 - e^{2i\mathrm{ArcCsc}[x^2]}]) + \mathrm{PolyLog}[2, e^{2i\mathrm{ArcCsc}[x^2]}])$$

obviously it is nonelementary, as discussed earlier. Let us take one more example

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[x^2]}{x^2} dx$$

and using Mathematica, we get

$$\mathrm{In}[29]{:}\ \mathrm{Integrate}[\frac{\mathrm{ArcCsc}[x^2]}{x^2}, x]$$

$$\mathrm{Out}[29]{:}\ -\frac{1}{x^3}(x^2\mathrm{ArcCsc}[x^2] + \frac{1}{\sqrt{1-\frac{1}{x^4}}}2(-1+x^4 + \sqrt{x^2}\sqrt{1-x^4}(-\mathrm{EllipticE}[\mathrm{ArcSin}[\sqrt{x^2}], -1] + \mathrm{EllipticF}[\mathrm{ArcSin}[\sqrt{x^2}], -1])))$$

which is nonelementary. Let us take some arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2$+1. Then from (1.5), we get

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[x^2 + 2x]}{2x^2 + 1} dx$$

In this case the Mathematica doesn't produce any output although for some integrands, it gives output in either elementary or in nonelementary form. So let us simplify it for

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[x^3]}{x^2} dx$$

Using Mathematica, we get

$$\mathrm{In}[30]{:}\ \mathrm{Integrate}[\frac{\mathrm{ArcCsc}[x^3]}{x^2}, x]$$

$$\mathrm{Out}[30]{:}\ \frac{1}{5\sqrt{1-\frac{1}{x^6}}x^4}(15 - 15x^6 - 5\sqrt{1-\frac{1}{x^6}}x^3\mathrm{ArcCsc}[x^3] + 6x^6\sqrt{1-x^6}\mathrm{Hypergeometric2F1}[\frac{1}{2}, \frac{5}{6}, \frac{11}{6}, x^6])$$

which gives integral in Hypergeometric function form, which is obviously nonelementary. Let us verify it for one more example as

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[x^3]}{x^3} dx$$

We get from Mathematica code that

$$\text{In[31]: Integrate}[\frac{\text{ArcCsc}[x^3]}{x^3}, x]$$

$$\text{Out[31]:} -\frac{1}{4x^5}(2x^3\text{ArcCsc}[x^3] + \frac{1}{\sqrt{1-\frac{1}{x^6}}}(3(-1 + x^6) + (-1)^{2/3}3^{3/4}(x^3)^{2/3}\sqrt{(-1)^{5/6}(-1+(x^3)^{2/3})}\sqrt{1+(x^3)^{2/3}+(x^3)^{4/3}}. (\sqrt{3}\text{EllipticE}[\text{ArcSin}[\frac{\sqrt{-(-1)^{5/6} - i(x^3)^{2/3}}}{3^{1/4}}], (-1)^{1/3}] + (-1)^{5/6}\text{EllipticF}[\text{ArcSin}[\frac{\sqrt{-(-1)^{5/6} - i(x^3)^{2/3}}}{3^{1/4}}], (-1)^{1/3}])))$$

which is in Elliptic function form, and so it is nonelementary. Again for one more example

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[\mathrm{x}^4]}{\mathrm{x}^4}\mathrm{dx}$$

We get using Mathematica that

$$\text{In[32]: Integrate}[\frac{\text{ArcCsc}[x^4]}{x^4}, x]$$

$$Out[32]: \frac{1}{45\sqrt{1-\frac{1}{x^8}}x^7}(20 - 20x^8 - 15\sqrt{1-\frac{1}{x^8}}x^4\text{ArcCsc}[x^4] + 4x^8\sqrt{1-x^8}\text{Hypergeometric2F1}[\frac{1}{2}, \frac{5}{8}, \frac{13}{8}, x^8])$$

which is obviously nonelementary. Let us take one more example

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[\mathrm{x}^4]}{\mathrm{x}^3}\mathrm{dx}$$

and using Mathematica, we get

$$\text{In[33]: Integrate}\left[\frac{\text{ArcCsc}[x^4]}{x^3}, x\right]$$

$$Out[33]: -\frac{1}{2x^6}(x^4\text{ArcCsc}[x^4] + \frac{1}{\sqrt{1-\frac{1}{x^8}}}2(-1+x^8 + \sqrt{x^4}\sqrt{1-x^8}(-\text{EllipticE}[\text{ArcSin}[\sqrt{x^4}], -1] + \text{EllipticF}[\text{ArcSin}[\sqrt{x^4}], -1])))$$

which is nonelementary.

Thus we can state that for f(x) = $\mathrm{cosec}^{-1}x$ and for polynomial h(x) of degree greater than or equal to one, the integral (1) is elementary for n = 1, m = 0 and nonelementary for other cases i.e., it is seen that the integral of the form

$$\mathrm{I} = \int \frac{\mathrm{cosec}^{-1}[\mathrm{x}^\mathrm{n}]}{\mathrm{x}^\mathrm{m}}\mathrm{dx}$$

are nonelementary except for n = 1, m = 0 (which is beyond our assumptions). No result comes out from Mathematica for complex polynomial having more than one term in it i.e. both the polynomials should have only one leading term with the highest degree.

Case-VI: When we take $f(x) = sec^{-1}x$, the integral (1) becomes

$$\mathrm{I} = \int \frac{\sec^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}}\mathrm{dx} \qquad (1.6)$$

Taking special case for both g(x) = h(x) = x, we get from (1.6)

$$I = \int \frac{sec^{-1}x}{x}dx$$

Applying Mathematica, we get

$$\text{In[34]: Integrate}[\frac{\text{ArcSec}[x]}{x}, x]$$

$$\text{Out[34]:} \frac{1}{2}i\text{ArcSec}[x]^2 - \text{ArcSec}[x]\text{Log}[1 + e^{2i\text{ArcSec}[x]}] + \frac{1}{2}i\text{PolyLog}[2, -e^{2i\text{ArcSec}[x]}]$$

which is in the form of polylogarithm and thus nonelementary.

If we consider another case like g(x) = x and h(x) = 1, although it is beyond our assumption. Then from (1.6), we have

$$\mathrm{I} = \int \frac{\sec^{-1}[\mathrm{x}]}{1}\mathrm{dx} = \int \sec^{-1}[\mathrm{x}]\mathrm{dx}$$

Integrating using Mathematica, it we get

$$\text{In[35]: Integrate[ArcSec}[x], x]$$

$$\text{Out[35]:} x\text{ArcSec}[x] - \frac{\sqrt{-1+x^2}\text{Log}[2(x+\sqrt{-1+x^2})]}{\sqrt{1-\frac{1}{x^2}}x}$$

which is elementary. Let us take another case when g(x) = x^2 and h(x) = 2x. Then from (1.6), we get

$$\mathrm{I} = \int \frac{\sec^{-1}[\mathrm{x}^2]}{2\mathrm{x}}\mathrm{dx}$$

Using Mathematica for integration, we get

$$\text{In}[36]\text{: Integrate}[\frac{\text{ArcSec}[x^2]}{2*x},x]$$

$$\text{Out}[36]\text{:}\frac{1}{8}i(\text{ArcSec}[x^2](\text{ArcSec}[x^2]+2i\text{Log}[1+e^{2i\text{ArcSec}[x^2]}])+\text{PolyLog}[2,-e^{2i\text{ArcSec}[x^2]}])$$

obviously it is nonelementary, as discussed earlier. Let us take one more example

$$\mathrm{I}=\int\frac{\sec^{-1}[\mathrm{x}^2]}{\mathrm{x}^2}\mathrm{dx}$$

and using Mathematica, we get

$$\text{In}[37]\text{: Integrate}[\frac{\text{ArcSec}[x^2]}{x^2},x]$$

$$\text{Out}[37]\text{:}\frac{1}{x^3}(-x^2\text{ArcSec}[x^2]+\frac{1}{\sqrt{1-\frac{1}{x^4}}}2(-1+x^4+\sqrt{x^2}\sqrt{1-x^4}(-\text{EllipticE}[\text{ArcSin}[\sqrt{x^2}],-1]+\text{EllipticF}[\text{ArcSin}[\sqrt{x^2}],-1])))$$

which is nonelementary. Let us take some arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2$+1. Then from (1.6), we get

$$\mathrm{I}=\int\frac{\sec^{-1}[\mathrm{x}^2+2\mathrm{x}]}{2\mathrm{x}^2+1}\mathrm{dx}$$

In this case the Mathematica doesn't produce any output although for some integrands, it gives output in either elementary or in nonelementary form. So let us simplify it for

$$\mathrm{I}=\int\frac{\sec^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\mathrm{dx}$$

Using Mathematica, we get

$$\text{In}[38]\text{: Integrate}[\frac{\text{ArcSec}[x^3]}{x^2},x]$$

$$\text{Out}[38]\text{:}\frac{1}{5\sqrt{1-\frac{1}{x^6}}x^4}(15(-1+x^6)-5\sqrt{1-\frac{1}{x^6}}x^3\text{ArcSec}[x^3]-6x^6\sqrt{1-x^6}\text{Hypergeometric2F1}[\frac{1}{2},\frac{5}{6},\frac{11}{6},x^6])$$

which gives integral in Hypergeometric function form, which is obviously nonelementary. Let us verify it for one more example as

$$\mathrm{I}=\int\frac{\sec^{-1}[\mathrm{x}^3]}{\mathrm{x}^3}\mathrm{dx}$$

We get using Mathematica that

$$\text{In}[39]\text{: Integrate}[\frac{\text{ArcSec}[x^3]}{x^3},x]$$

$$\text{Out}[39]\text{:}\frac{1}{4x^5}(-2x^3\text{ArcSec}[x^3]+\frac{1}{\sqrt{1-\frac{1}{x^6}}}(3(-1+x^6)+(-1)^{2/3}3^{3/4}(x^3)^{2/3}\sqrt{(-1)^{5/6}(-1+(x^3)^{2/3})}\sqrt{1+(x^3)^{2/3}+(x^3)^{4/3}}.(\sqrt{3}\text{EllipticE}[\text{ArcSin}[\frac{\sqrt{-(-1)^{5/6}-i(x^3)^{2/3}}}{3^{1/4}}],(-1)^{1/3}]+(-1)^{5/6}\text{EllipticF}[\text{ArcSin}[\frac{\sqrt{-(-1)^{5/6}-i(x^3)^{2/3}}}{3^{1/4}}],(-1)^{1/3}])))$$

which is in Elliptic function form, and so it is nonelementary. Again for one more example

$$\mathrm{I}=\int\frac{\sec^{-1}[\mathrm{x}^4]}{\mathrm{x}^4}\mathrm{dx}$$

We get using Mathematica code that

$$\text{In}[40]\text{: Integrate}[\frac{\text{ArcSec}[x^4]}{x^4},x]$$

$$Out[40]\text{:}\frac{1}{45\sqrt{1-\frac{1}{x^8}}x^7}(20(-1+x^8)-15\sqrt{1-\frac{1}{x^8}}x^4\text{ArcSec}[x^4]-4x^8\sqrt{1-x^8}\text{Hypergeometric2F1}[\frac{1}{2},\frac{5}{8},\frac{13}{8},x^8])$$

which is nonelementary. Let us take one more example

$$\mathrm{I}=\int\frac{\sec^{-1}[\mathrm{x}^4]}{\mathrm{x}^3}\mathrm{dx}$$

and using Mathematica, we get

$$\text{In}[41]\text{: Integrate}\left[\frac{\text{ArcSec}[x^4]}{x^3},x\right]$$

$$Out[41]\text{:}\frac{1}{2x^6}(-x^4\text{ArcSec}[x^4]+\frac{1}{\sqrt{1-\frac{1}{x^8}}}2(-1+x^8+\sqrt{x^4}\sqrt{1-x^8}(-\text{EllipticE}[\text{ArcSin}[\sqrt{x^4}],-1]+\text{EllipticF}[\text{ArcSin}[\sqrt{x^4}],-1])))$$

which is nonelementary.

Thus we can state that for f(x) = $\sec^{-1}x$ and for polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is elementary for n = 1, m = 0 and nonelementary for other cases i.e., it is seen that the integral of the form

$$\mathrm{I} = \int \frac{\sec^{-1}[x^n]}{x^m} dx$$

are nonelementary except for n = 1, m = 0 (which is beyond our assumptions). In all cases, both the polynomials should have only one leading coefficient term with highest degree.

From above discussion, we can conclude that the antiderivative (1) has the same nature of integral for the pair wise inverse trigonometric functions ($\sin^{-1}x$, $\cos^{-1}x$), ($\tan^{-1}x$, $\cot^{-1}x$) and ($\mathrm{cosec}^{-1}x$, $\sec^{-1}x$). The common nature for the integrals for all inverse trigonometric functions is that the integral (1) is nonelementary for g(x) = h(x) = x and h(x) = 2x, g(x) = x^2, and elementary for g(x) = x, h(x) = 1. For g(x) = x^3, h(x) = x^2 and f(x) = $\sin^{-1}x$, $\cos^{-1}x$, $\mathrm{cosec}^{-1}x$, $\sec^{-1}x$, the integral (1) is nonelementary but for f(x) = $\mathrm{tann}^{-1}x$ and $\cot^{-1}x$, it is elementary.

The integral (1) is always elementary for f(x) = $\tan^{-1}x$, $\cot^{-1}x$ for all polynomials g(x), h(x) of degree greater than or equal to two. Whereas for f(x) = $\sin^{-1}x$, $\cos^{-1}x$, $\mathrm{cosec}^{-1}x$, $\sec^{-1}x$, the integral (1) is always nonelementary for higher degree polynomials g(x), h(x) of degree greater than or equal to two. In all cases each polynomial has only one term of highest degree term. For mixed polynomial g(x) = x^2+2x, h(x) = 2x + 1, the mathematica7 code doesn't produce any result for all cases. Obviously then for complex polynomials, we cannot decide its nature using computer software Mathematica.

5. Conclusion

We can conclude and propound the conjecture for the antiderivative (1) that it is always elementary for f(x) = $\tan^{-1}x$, $\cot^{-1}x$ and nonelementary for f(x) = $\sin^{-1}x$, $\cos^{-1}x$, $\mathrm{cosec}^{-1}x$, $\sec^{-1}x$ for all polynomials g(x), h(x) of degree greater than or equal to two. The only condition is imposed on all polynomials is that it should have only one term with highest degree.

6. Limitations

The polynomial bears the minimum number of terms, in particular only one term with highest degree. No coefficient has been taken to make the calculation easy. Whenever we took the polynomial in complex form containing more than one term, the Mathematica doesn't give output always. This is the big limitation of the study.

7. Future Scope of Research

The paper has been discussed for the polynomials of lower degree with minimum number of terms. A lot of scope is available for further research by taking general forms of polynomials of higher degrees.

8. Acknowledgment

This paper is based on the third conjecture propounded by Yadav & Sen (2008, 2012) in their doctoral thesis. A paper based on this conjecture with different f(x) has already been submitted and is under review in a journal, which has been propounded on the base of inverse hyperbolic functions.

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