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Multi-Echelon Supply Chain Inventory Model for Deteriorating Items Under the Effect of Inflation and Preservation Technology

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Abstract

In the present model we have developed multi-echelon supply chain model for deteriorating items with exponential time dependent demand and non-instantaneous deterioration. Preservation technology was implemented to control the deterioration of the manufacturer items which provide extra protection items. Entire study carried out under the impact of inflation. In the present study we consider the two suppliers involved in the supply chain for supplying raw materials to the manufacturer. The manufacturer produces a random proportion of deteriorating items which are reworked after daily production. Items sold out in lot to other markets after completing the rework process. Retailer sells finished items in various markets according to seasons. Numerical examples with optimal solution present the validity of the model. Finally, sensitivity analysis and graphical representation have been shown for the analysis.

Keywords: Multi-echelon, Optimization, deterioration, inventory, inflation

1. Introduction

A Multi-Echelon System is a hierarchical inventory management structure that consists of multiple levels or echelons of inventory system. Every echelon shows a different stage in the supply chain, such as production, distribution, warehousing, and retail. There are some important keywords of MES which are such as- (a) Multiple Echelons: MES consists of multiple levels of inventory control, representing a different stage in the supply chain. (b) Inventory Management: Each echelon manages its own inventory, including inventory levels, replenishment, and control. (c) Coordination and Optimization: MES aims to coordinate and optimize inventory management across all echelons to minimize costs, reduce stock outs, and improve customer service. Multi-echelon system play a significant role in the inventory system and its provides the various benefits which are as- (a) Reduced Inventory Costs: MES helps minimize inventory costs by optimizing inventory levels and reducing waste. (b) Improved Inventory Management: MES enables better inventory management through coordinated control and optimization. (c) Reduced Stock outs: MES helps reduce stock outs by ensuring that inventory is available when needed. (d) Improved Customer Service: MES enables better customer service by providing accurate and timely inventory information.

1.1 Relation between deterioration and multi-echelon system

Multi-echelon systems and deterioration of items are closely related. As items move through various echelons of the supply chain, they are exposed to various environmental factors that can accelerate deterioration. To minimize the deterioration of the items in multi-echelon such as- (a) Proper Handling and Transportation: Ensure careful handling and transportation of items between echelons. (b) Suitable Storage Conditions: Maintain optimal storage conditions at each echelon. (c) Regular Inventory Turnover: Monitor and control inventory turnover rates across echelons. (d) Impassive Quality Control: Implement robust quality control measures at each echelon. For the multi-echelon system and deterioration various prominent researchers have done the work in such manners.

1.2 Two-echelon supply chain and deterioration related review work

Adak and Mahapatra [1] developed two-layer supply chain models for manufacturers, retailers and probabilistic deteriorating items in the fuzzy environment. They worked with time and reliability dependent demand. Barman and Das [2] optimized the total cost for deteriorating items with bi-level trade credit policy and demand function was dependent on the time and ramp type. Iraj et al. [3] proposed two-echelon inventory model with the expiration date of products. They considered fixing production dependent demand function for deteriorating items. Costa et al. [4] investigated two-echelon supply chain problems inspired by real-word production. They focused on the unreliable manufacturing process. Further, deterioration and preservation technology related significant research work done by various prominent researchers and authors in the such as- deterioration means vaporization, decay, waste etc. work as Bhawaria and Rathore [5], Bhawaria and Rathore [6], Bhawaria et al. [7], Rathore et al. [8], Utama et al. [9], Kaushik [10].

1.3 Three-echelon supply chain and deterioration related review work

The Three-Echelon Supply Chain System is a hierarchical supply chain structure consisting of three different echelons which are works as- (a) Manufacturer Echelon: This is the first echelon, responsible for producing or supplying raw materials, finished goods. (b) Wholesale Echelon: The second echelon, which receives items from the manufacturer and stores, distributes, and delivers them to the next level. (c) End-Customer Echelon: The final echelon, where products are sold directly to end-customers. But there are some significant profits of this process are given below as- (a) Improved supply chain efficiency and effectiveness (b) Enhanced inventory management and control (c) Better customer service and satisfaction (d) Reduced total costs and improved profitability. Further, for the three-echelon and deteriorating items many researchers have done the work such as- Mashud et al. [11] formulated three-echelon supply chain model for deteriorating items and they used green technology to control the deterioration of the items. This entire research work main aim was to reduce carbon emissions and control the deterioration. Wang et al. [12] determined the pricing policy and green efforts for the three-echelon supply chain system. Mondal et al. [13] established payment policy for three-echelon system. They focused on the advertisement demand function and multiple delivery policy. Navarro et al. [14], Pokorny and Fiala [15] established three echelon supply chain models for deteriorating items. They specially focused on the constant deterioration and manufacturing process.

1.4 Four-echelon supply chain and deterioration related review work

Four-Echelon Supply Chain System

The Four-Echelon supply chain system (SCS) is a hierarchical supply chain structure consisting of four different levels. (a) Manufacturer Echelon: This is the first echelon, responsible for producing or supplying raw materials, components, or finished goods. (b) Wholesaler Echelon: The second echelon, which receives products from the manufacturer and stores, distributes, and delivers them to the next level. (c) Regional Distribution Center (RDC) Echelon: The third echelon, which receives products from the distributor/wholesaler and stores, distributes, and delivers them to the final level. (d) End-Customer Echelon: The fourth and final echelon, where products are sold directly to end-customers. Further, review work done by prominent researchers which is given in table-1.

Table 1 literature survey and research gap

Authors Name	**Demand Type**	**PTC**	**Inflation**	**Multi-Echelon**	**Model Type**
Sebatjane & Adetunji [16]	Quality based	No	No	Four-echelon	Inventory model
Ali et al. [17]	Deterministic	No	No	Four-echelon	EOQ
Correia & Melo [18]	Deterministic	No	No	Three-echelon	EOQ
Khalfi et al. [19]	Price dependent	No	No	Four-echelon	inventory
Saeedi et al. [20]	Stochastic	No	No	Four-echelon	-
This Paper	Time dependent	Yes	Yes	Multi-echelon	SCM

2. Assumptions and Notations

2.1 Notations

Notations	**Description**
$I_{1S}(t)$	Main supplier: inventory of raw items at t.
$I_{2S}(t)$	Secondary supplier: inventory of raw items at time t.
$I_{M_i}(t)$	Manufacturer: inventory of finished items at time t.
$I_{R_K}(t)$	Retailers: inventory of good quality items at time (T_{K-1}, T_K).
P	Replenishment rate of manufacturer.
IC_{1S}	Inventory cost for main supplier.
IC_{2S}	Inventory cost for secondary supplier.
τ_θ	Deterioration rate of items.
PTC	Preservation technology cost.
ξ	Preservation technology cost parameter.
R	Whole order quantity (OQ) of manufacturer to key supplier.
A_{1S}	Set up cost for first supplier.
A_{2S}	Set up cost for second supplier.
r	Inflation rate
ε	Random time.
C_{1HS}	Holding cost per unit per unit time at main supplier level.
C_{2HS}	Holding cost per unit per unit time at secondary supplier level.
μ	Demand parameter.
P_1	Deteriorating items remanufacturing rate of manufacturer.
D_r	Demand of odd dealer.
D_{C_K}	Demand of consumers.
t_R	Rework time of deteriorating items.
t_1	Manufacturer production run-time.
T	Total cycle length.
n	The number of selling seasons for retailers.
m	Positive no. $m \in [0, n]$.
C_{HM}	Holding cost/unit/unit time for good quality items.
C'_{HM}	Holding cost per unit per unit time for deteriorating goods.
C''_{HM}	Holding cost/unit/unit time for reworked items.
r_M	Rework cost/deteriorating items of manufacturer.

2.2 Assumptions

- Present model developed for single item.
- Manufacturer production rate is fixed and is greater than demand of odd dealer to neglect the shortages.
- The rate of production of deteriorating products is random and scrape products are not manufactured during daily production.
- Shortages are not allowed.
- Reworked goods are trade to another market in a lot after complete of remanufacture.
- In the every production run, rework start just after the end of regular manufacturing process and finished products are not as good as the real quality.
- The demand function is time dependent given as $D(t) = a + be^{\mu t}$ where $a > 0, b > 0, \mu > 0.$ and equal for every market.
- The holding cost of the items is parabolic function of the time and represented by $(f + gt^2)$. Where $f > 0, g > 0.$
- Retailers vend their items in different market with different selling price.
- After the random time main supplier can face disruption due to lake of raw material in the market. Then the other supplier, totally reliable but too much expensive than main supplier is chosen for supply of raw items.

- The deterioration rate of the items is fix and defined as constant rate$\tau_\theta = (\theta - m(\xi))$, where$m(\xi) = e^{-b\xi}$.

3. Mathematical Model Formulation

In this supply chain inventory model we consider two suppliers, supplier one is manufacturer and second is retailer in various market with new seasons. In the initial stage the major supplier procure raw material from other supplier and delivers at the fix rate to producer. But first supplier can face supply disruption problem after a random time (ε) cause of transportation and other things. Then the producer is helpless to order the raw item from the second supplier. Producer produces the products with fix rate (P). In the manufacturing process deteriorating goods are manufactured at the random rate $\tau_\theta$unit/unit time. After the steady manufacturing of deteriorating goods are reworked at the any cost. The governing differential equations are follows for supplier, manufacturer and retailer.

3.1. Mathematical model for supplier

Case 3.1.1 when $\boldsymbol{\varepsilon \le t_1}$

$$\frac{dI_{1S}(t)}{dt} = -P \qquad 0 \le t \le \varepsilon \tag{1}$$

$$\frac{dI_{2S}(t)}{dt} = -P \qquad \varepsilon \le t \le t_1 \tag{2}$$

Solution of equations (1) and (2) with boundary conditions $I_{1S}(0) = R, I_{2S}(\varepsilon) = R - P\varepsilon$

$$I_{1S}(t) = R - Pt, \tag{3}$$

$$I_{2S}(t) = R - Pt, \tag{4}$$

The condition$I_{2S}(t_1) = 0$, $t_1 = \frac{R}{P}$. (5)

Cost calculation for supplier

- **Inventory cost for main supplier**

$$IC_{1S} = C_{1S}\int_0^{\varepsilon} e^{-rt}(I_{1S})dt \tag{6}$$

$$IC_{1S} = C_{1S}\left[\frac{R}{r}(1 - e^{-r\varepsilon}) - P(\frac{1}{r^2}(1 - e^{-r\varepsilon}) + \frac{\varepsilon e^{-r\varepsilon}}{r})\right] \tag{7}$$

- **Inventory cost for secondary supplier**

$$IC_{2S} = C_{2S}\int_{\varepsilon}^{t_1} e^{-rt}(I_{2S})(f + gt^2)dt \tag{8}$$

$$IC_{2S} = C_{2S}\left[\frac{RF}{r}\{e^{-r\varepsilon} - e^{-rt_1}\} + \frac{gR}{r}\left\{e^{-rt_1}\left(\frac{2t_1}{r} - \frac{2}{r^3} - t_1^2\right) - e^{-r\varepsilon}\left(\frac{2\varepsilon}{r} - \frac{2}{r^3} - \varepsilon^2\right)\right\} - \frac{fP}{r}\left\{e^{-r\varepsilon}\left(\varepsilon + \frac{1}{r}\right) - e^{-rt_1}\left(\frac{1}{r} + t_1\right)\right\} - \frac{Pg}{r}\left\{\frac{3}{r}\left(e^{-rt_1}\left(\frac{2t_1}{r} - \frac{2}{r^3} - t_1^2\right) - e^{-r\varepsilon}\left(\frac{2\varepsilon}{r} - \frac{2}{r^3} - \varepsilon^2\right)\right) + \varepsilon^3 e^{-r\varepsilon} - t_1^3 e^{-rt_1}\right\}\right] \tag{9}$$

- **Preservation Technology Cost**

$$PTC = \int_0^{t_1} e^{-rt}\xi dt \tag{10}$$

$$PTC = \frac{\xi}{r}[1 - e^{-rt_1}] \tag{11}$$

Now the total cost for main supplier is given below

$$TC_{1S} = [setup\ cost + Inventory\ cost + Purchasing\ cost]$$

$$TC_{1S} = [A_{1S} + RC_{1S} + IC_{1S}]$$

Now the total cost for secondary supplier is given below

$$TC_{2S} = [setup\ cost + Inventory\ cost + Purchasing\ cost + PTC]$$

$$TC_{2S} = [A_{1S} + (R - P\varepsilon)C_{2S} + IC_{2S} + PTC]$$

Case 3.1.2 when $\boldsymbol{\varepsilon \ge t_1}$

$$\frac{dI_{1S}(t)}{dt} = -P \qquad 0 \le t \le t_1 \tag{12}$$

Solution of equation (12) with the boundary condition is $I_{1S}(0) = R$.

$$I_{1S}(t) = R - Pt, \tag{13}$$

Now $I_{1S}(t_1) = 0$ then $t_1 = \frac{R}{P}$. (14)

Inventory cost for main supplier

$$IC_{3S} = C_{1HS}\int_0^{t_1} e^{-rt}I_{1S}(t)dt \tag{15}$$

$$IC_{3S} = -C_{1HS}\left[\frac{R}{r}(e^{-rt_1} - 1) + \frac{P}{r}\left\{e^{-rt_1}\left(\frac{1}{r} + t_1\right) + \frac{1}{r}\right\}\right] \tag{16}$$

The total cost for main supplier

$$TCS_1 = [A_{1S} + RC_{1S} + IC_{3S}] \tag{17}$$

And the total cost for second supplier is zero.

The total cost for main supplier

$$TCS_1 = [A_{1S} + RC_{1S} + IC_{1S} + (R - P\varepsilon)C_{2S} + IC_{2S} + PTC + R_{1S}C_{1S} + IC_{3S}] \tag{18}$$

The total cost for secondary supplier

$$TCS_1 = [A_{2S} + (R - P\varepsilon)C_{2S} + IC_{2S}] \tag{19}$$

3.2 Manufacturer Cost

The manufacturer's good quality items inventory level at the time can be represented by the following differential equations.

$\frac{dI_{1m}(t)}{dt} = (1-\tau_\theta)P - (a + be^{\mu t}) \quad 0 \le t \le t_1$ (20)

$\frac{dI_{2m}(t)}{dt} = -(a + be^{\mu t}) \quad t_1 \le t \le T$ (21)

Solution of the equation (20) and (21) with boundary conditions- $I_{1m}(0) = 0$, $I_{1m}(t_1) = I_{2m}(t_1)$ and $I_{2m}(T) = 0$.

$I_{1m}(t) = (1-\tau_\theta)Pt - \left(at + \frac{be^{\mu t}}{\mu}\right) + \frac{b}{\mu}$ (22)

$I_{2m}(t) = a(T-t) + \frac{b}{\mu}(e^{\mu T} - e^{\mu t})$ (23)

Now applying the condition $I_{1m}(t_1) = I_{2m}(t_1)$ we have

$T = \frac{1}{a+b}[(1-\tau_\theta)Pt_1]$ (24)

- **Inventory cost of good quality items**

$$IC_M = C_{HM}\left[\int_0^{t_1} e^{-rt} I_{1m}(t)(f + gt^2)dt + \int_{t_1}^{T} e^{-rt} I_{2m}(t)(f + gt^2)dt\right] \quad (25)$$

$$IC_M = C_{HM}\Big[fP\left\{-e^{-rt_1}\left(\frac{t_1}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}\right\} + fP\tau_\theta\left\{\frac{e^{-rt_1}}{r}\left(t_1 + \frac{1}{r}\right) - \frac{1}{r^2}\right\} - fa\left\{\frac{e^{-rt_1}}{r}\left(t_1 + \frac{1}{r}\right) - \frac{1}{r^2}\right\} - \frac{fb}{\mu(\mu+r)}\{1 - e^{-t_1(r+\mu)}\} + \frac{fb}{\mu r}\{1 - e^{-rt_1}\} + \{gP - ga - \tau_\theta Pg\}\left\{\frac{t_1^3 e^{-rt_1}}{-r} + \frac{3}{r}\left\{\frac{t_1^2 e^{-rt_1}}{-r} - \frac{2}{r}\left(\frac{e^{-rt_1}}{r} + \frac{e^{-rt_1}}{r^2}\right)\right\} + \frac{6}{r^4}\right\} - \frac{g}{\mu}\left\{\frac{t_1^2 e^{t_1(\mu-r)}}{\mu-r} - \frac{2}{\mu-r}\left(\frac{t_1 e^{t_1(\mu-r)}}{\mu-r} - \frac{e^{t_1(\mu-r)}}{(\mu-r)^2}\right) + \frac{2}{(\mu-r)^3}\right\} + \frac{gb}{\mu}\left\{\frac{t_1^2 e^{-rt_1}}{-r} + \frac{2}{r}\left\{\frac{e^{-rt_1}}{-r} - \frac{e^{-rt_1}}{r^2}\right\} + \frac{2}{r^3}\right\}\Big] \quad (26)$$

The manufacturers' items level of deteriorating items at time t presented by the equations.

$\frac{dI_{1d}(t)}{dt} = \tau_\theta \quad 0 \le t \le t_1$ (27)

$\frac{dI_{2d}(t)}{dt} = t_1\tau_\theta - P_1 \quad t_1 \le t \le T$ (28)

$\frac{dI_{3d}(t)}{dt} = P_1 \quad t_1 \le t \le T$ (29)

Solution of equation (27) with condition $I_{1d}(0) = 0, I_{1d}(t_1) = t_1\tau_\theta$ and $I_{1d}(t_1) = 0$.

$I_{1d}(t) = \tau_\theta t \quad 0 \le t \le t_1$ (30)

$I_{2d}(t) = t_1\tau_\theta(t - t_1 + 1) - P_1(t - t_1)$ (31)

$I_{3d}(t) = P_1(t - t_1)$ (32)

Rework time for deteriorating items $t_R = \frac{t_1\tau_\theta}{P_1}$ (33)

Inventory cost for deteriorating items

$$IC_D = \left[C'_{HM}\left\{\int_0^{t_1} e^{-rt}\tau_\theta(f + gt^2)dt + \int_{t_1}^{t_1+t_R} e^{-rt}(t_1\tau_\theta - P_1)(f + gt^2)dt\right\} + C''_{HM}\left(\int_{t_1}^{t_1+t_R} e^{-rt}(P_1)(f + gt^2)dt\right)\right] \quad (34)$$

$$IC_D = \Bigg[C'_{HM}\Bigg\{\tau_\theta\left\{\frac{f}{r}(1 - e^{-rt_1}) - \frac{gt_1^2 e^{-rt_1}}{r} - \frac{2}{r^2}\left\{t_1 e^{-rt_1} - \frac{e^{-rt_1}}{r} - \frac{1}{r}\right\}\right\}\Bigg\} + \{(\tau_\theta t_1 - P_1) + P_1 C''_{HM}\}\left\{\frac{f}{r}\left(e^{-rt_1} - e^{-r(t_1+t_R)}\right) + \frac{g}{r}\{t_1{}^2 e^{-rt_1} - e^{-r(t_1+t_R)}(t_1 + t_R)^2\} + \frac{2g}{r^2}\left\{e^{-rt_1}\left(t_1 + \frac{1}{r}\right) - e^{-r(t_1+t_R)}\left(t_1 + t_R + \frac{1}{r}\right)\right\}\right\}\Bigg] \quad (35)$$

Total rework cost for deteriorating items

$R_C = \tau_\theta t_1 r_M$ (36)

- **Preservation technology cost**

$PTC = \int_0^{t_1+t_R} \xi e^{-rt} dt$ (37)

$PTC = -\frac{\xi}{r}\left[e^{-r(t_1+t_R)} - \frac{1}{r}\right]$ (38)

- ***Total cost***

$TC = [SC + PTC + PC + IC + RC]$ (39)

3.3 Retailer's individual cost

$\frac{dI_{R_K}(t)}{dt} = (a + be^{\mu t})_R - (a + be^{\mu t})_{C_K}$,

With $I_{R_1}(0) = 0$ and $I_{R_K}(T_{K-1}) = I_{R_{K-1}}(T_{K-1})$

$, T_{K-1} \le t \le T_K, K = 1,2,3,\dots,(M-1),$ (40)

$\frac{dI_{R_M}{}^{-}(t)}{dt} = -(a + be^{\mu t})_{C_M} with\ I^{-}_{R_M}(T_{M-1}) = I_{R_{M-1}}(T_{M-1}), T_{M-1} \le t \le T$ (41)

$\frac{dI^{+}_{R_M}(t)}{dt} = -(a + be^{\mu t})_{C_M},\ with\ I^{+}_{R_M}(T) = I^{-}_{R_M}(T), T \le t \le T_M,$ (42)

$\frac{dI_{R_K}(t)}{dt} = -(a + be^{\mu t})_{C_K}, \quad with\ I_{R_{M+1}}(T_M) = I^{+}_{R_M}(T_M), I_{R_n}(T_n) = 0, and\ I_{R_K}(T_{K-1}) = I_{R_{K-1}}(T_K)$
$T_{K-1} \le t \le T_K, K = M+1, M+2, M+3, \dots, n$ (43)

Solution of the above equations (40) to (43) with the boundary conditions

$I_{R_K}(t) = \left(aT_{C_K} - be^{\mu T_{C_{K-1}}}\right) + \sum_{i=1}^{K}\left(\left(aT_{C_K} - be^{\mu T_{C_{K-1}}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T_{C_{K-1}}}\right)_{C_{i-1}}\right)T_{i-1}\ K = 1,2,3,\dots,(M-1)$ (44)

$$I_{R_M}{}^{-}(t) = \left(aT_{C_K} - be^{\mu T c_{K-1}}\right) + \sum_{i=1}^{M}\left(\left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_{i-1}}\right)T_{i-1}$$

$T_{M-1} \leq t \leq T$ (45)

$I_{R_M}{}^{+}(t) = \left(-aT_{C_M} + be^{\mu T c_n}\right) - \sum_{i=M+1}^{n}\left(\left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_{i-1}}\right)T_{i-1}\ T \leq t \leq T_M$ (46)

$$I_{R_K}(t) = \left(-aT_{C_M} + be^{\mu T c_n}\right) - \sum_{i=K+1}^{n}\left(\left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_{i-1}}\right)T_{i-1}$$

$T_{C_{K-1}} \leq t \leq T_K$, $K = M+1, M+2, \ldots, n$ (47)

Now the condition $I_{R_M}{}^{+}(M) = I_{R_M}{}^{-}(M) = D_{C_n}T_{n.}$

Now the inventory calculation for the retailer

- **Inventory in the interval $[T_{K-1}, T_K]$, where $K = 1, 2, 3, \ldots, (M-1)$ is**

$$I_{R_K} = \int_{T_{K-1}}^{T_K} I_{R_K}(t)dt$$

$I_{R_K}(t) = [\left(aT_{C_K} - be^{\mu T c_{K-1}}\right) + \sum_{i=1}^{K}\left(\left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_{i-1}}\right)T_{i-1}](T_K - T_{K-1})$ (48)

- **Inventory in the interval $[T_{M-1}, T]$**

$$I_{R_M}{}^{-}(t) = \int_{T_{M-1}}^{T} I_{R_M}{}^{-}(t)dt$$

$I_{R_M}{}^{-}(t) = \left[\left(aT_{C_K} - be^{\mu T c_{K-1}}\right) + \sum_{i=1}^{M}\left(\left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_{i-1}}\right)T_{i-1}\right](T - T_{M-1})$ (49)

- **Inventory in interval $[T, T_M]$ is**

$$I_{R_M}{}^{+}(t) = \int_{T}^{T_M} I_{R_M}{}^{+}(t)dt$$

$I_{R_M}{}^{+}(t) = \left[\left(-aT_{C_M} + be^{\mu T c_n}\right) - \sum_{i=M+1}^{n}\left(\left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_{i-1}}\right)T_{i-1}\right](T_M - T)$ (50)

- **Inventory in the interval $[T_{K-1}, T_K]$, where $K = M+1, M+2, M+3, \ldots, n$**

$$I_{R_K}(t) = \int_{T_{K-1}}^{T_K} I_{R_K}(t)dt$$

$I_{R_K}(t) = \left[\left(-aT_{C_M} + be^{\mu T c_n}\right) - \sum_{i=K+1}^{n}\left(\left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_i} - \left(aT_{C_K} - be^{\mu T c_{K-1}}\right)_{C_{i-1}}\right)T_{i-1}\right](T_K - T_{K-1})$ (51)

- **Inventory cost for retailer**

$$IC_R = C_{HR}\left[\sum_{K=1}^{M-1} I_{R_K} + I_M^- + I_M^+ + \sum_{K=M+1}^{n} I_{R_K}\right]$$

$IC_R = \frac{C_{HR}}{2}\left[\sum_{K=1}^{n}(a + be^{\mu t})_{C_K}(T_K^2 - T_{K-1}^2)\right]$

(52)

- **Total cost**

$TC = [SC + PC + IC]$ (53)

4. Solution Process

To minimize the total cost we diff. TC (T, t_1,ξ) with respect to T, ξ, and t_1. And for optimum values necessary conditions are-

$$\frac{\partial TC(T, t_1, \xi)}{\partial T} = 0, \frac{\partial TC(T, t_1, \xi)}{\partial \xi} = 0, \frac{\partial TC(T, t_1, \xi)}{\partial t_1} = 0$$

$\det(H_1) > 0$, $\det(H_2) > 0$, $\det(H_3) > 0$, Hessian Matrix of the total cost is as follows.

$$\begin{bmatrix} \frac{\partial^2(TC)}{\partial \xi^2} & \frac{\partial^2(TC)}{\partial \xi \partial t_1} & \frac{\partial^2(TC)}{\partial \xi \partial T} \\ \frac{\partial^2(TC)}{\partial t_1 \partial \xi} & \frac{\partial^2(TC)}{\partial t_1^2} & \frac{\partial^2(TC)}{\partial t_1 \partial T} \\ \frac{\partial^2(TC)}{\partial T \partial \xi} & \frac{\partial^2(TC)}{\partial T \partial t_1} & \frac{\partial^2(TC)}{\partial T^2} \end{bmatrix}$$

4.1 Numerical Example

We have taken the appropriate values of the inventory parameters in the suitable units which are given as-

$$C_{1S} = 6568, f = 5600, R_{1S} = 500, A_{1S} = 778, r = 0.05, R = 679, \epsilon = 3790, g = 4500, P = 79900, C_{1HS} = 400, C_{2S} = 4598, C_{1S} = 6568.$$

Now the optimum values are-

$$T = 4.05199 \times 10^{21}, \quad t_1 = 6.111 \times 10^7, \quad \xi = 18,\ TC = 794791.1541.$$

4.2 Optimum solution:

n	$T_k \times 10^{21}$	$D_{C_K} \times 10^{21}$	ξ	$t_1 \times 10^7$	T$\times 10^{21}$	TC
1	$T_1 = 3.03465$	$D_{C_1} = 4.86239$	16.00	5.897	3.87481	797569.1823
2	$T_1 = 2.09856$	$D_{C_1} = 4.86239$	16.89	5.972	2.74845	797121.1639
	$T_2 = 3.09959$	$D_{C_2} = 4.94343$				
3	$T_1 = 3.02567$	$D_{C_1} = 4.86239$	17.12	5.973	2.89464	797111.7621
	$T_2 = 3.07856$	$D_{C_2} = 4.94343$				
	$T_3 = 3.09874$	$D_{C_3} = 5.02447$				
4	$T_1 = 3.04358$	$D_{C_1} = 4.86239$	18.00	6.111	4.05199	794791.1541
	$T_2 = 3.01231$	$D_{C_2} = 4.94343$				
	$T_3 = 2.99564$	$D_{C_3} = 5.02447$				
	$T_4 = 2.98453$	$D_{C_4} = 5.10551$				
5	$T_1 = 3.01122$	$D_{C_1} = 4.86239$	17.34	6.102	4.01732	794796.1423
	$T_2 = 2.93751$	$D_{C_2} = 4.94343$				
	$T_3 = 2.95467$	$D_{C_3} = 5.02447$				
	$T_4 = 2.87347$	$D_{C_4} = 5.10551$				
	$T_5 = 2.98173$	$D_{C_5} = 5.18655$				
6	$T_1 = 2.88761$	$D_{C_1} = 4.86239$	16.96	6.012	4.03879	794796.8123
	$T_2 = 2.78465$	$D_{C_2} = 4.94343$				
	$T_3 = 2.73628$	$D_{C_3} = 5.02447$				
	$T_4 = 2.65172$	$D_{C_4} = 5.10551$				
	$T_5 = 2.87698$	$D_5 = 5.18655$				
	$T_6 = 2.98763$	$D_{C_6} = 5.26759$				

4.3 Graphical Presentation

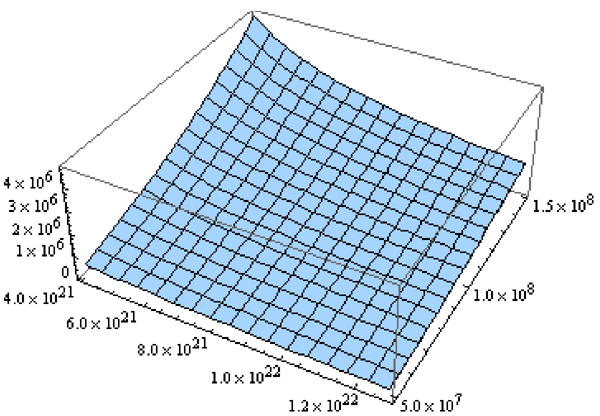

Fig. 1 Graph between T &t_1 w.r.t. TC

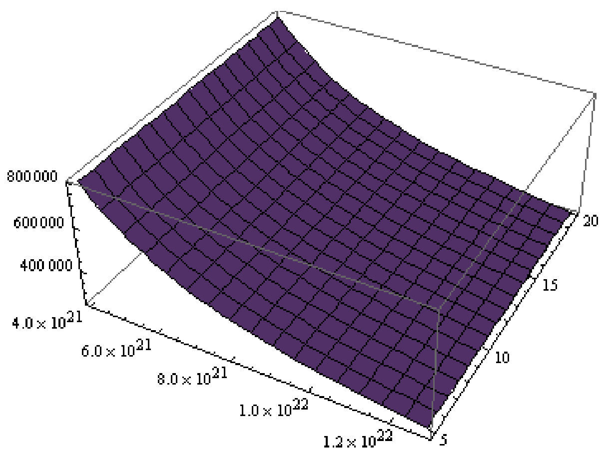

Fig. 2 Graph between T &ξ w.r.t. TC

Conclusion

In the present research work we established multi-echelon supply chain inventory model for deteriorating items. Preservation technology applied to control the deterioration of the items and it does provide extra protection to items. During the manufacturing time deteriorating items are manufactured at random rate and after manufacturing they are remanufactured and sold in different markets. But the retailer has fix demand of the items and sells them in different markets with different selling prices to get the maximum profit by the items. And the entire system is affected by the rate of inflation. Our numerical example we concluded that at n=4 the total cost of the system is optimized. In the future this model can be extended with different demand functions like- selling price, stock dependent demand, fuzzy triangular demand function. We described the following limitations of this model which are as- the stock out of condition in every stage of chain system is degraded.

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