Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 46 No. 9
September 2025

A Comprehensive Study Of Sustainable Future Of Fractional Fourier Transform

[1]**Abhisekh Shekhar,**

[1] Assistant Professor, Department of Mathematics, C.M.Science College,Darbhanga,Bihar,India-846004

Abstract: The Fractional Fourier transform (FRFT) is a relatively new linear transform that is a gen- eralisation of the classical Fourier transform (FT). FRFT can transform a particular signal to a unified time-frequency domain. In this study of sustainable future, we try to present a comprehensive discussion FRFT. Firstly, we provided definition of FRFT. Secondly, we offered a comprehensive theoretical research and technological studies that consisted of hardware implementation and software implementation. Thirdly, we presented a sustainable future on applications of FRFT to following fields: communication, encryption, optimal engineering, radiology, remotesensing, fractional calculus, fractional wavelet transform, pseudo-differential operator, pattern recognition, and image processing. It is hoped that this comprehensive study of sustainable Future will be useful to research scholars/faculties/scientists working on FRFT.

Keywords: Fractional Fourier transform; discrete fractional Fourier transform; time-frequenc analysis; fractional wavelet transform; signal processing; fractional calculus.

1. Introduction

The French mathematician Jean-Baptiste Joseph Fourier(1768-1830)introduced a major mathematical concept in basic sciences, known as the Fourier transform (FT), two hundred fifteen years ago. it was first used by him in a manuscript submitted to the Institute of France in 1807 [1] (J. B.J.Fourier, Thorie de la propagation de la chaleur dans les solides, Manuscrispt submitted to the Institute of France, 21 Dec. 1807). Fourier analysis did not become a standard tool in modern science until 1965, when Cooley and Tukey developed the 'fast Fourier transform' algorithm [2]. Because the traditional Fourier transform (FT) decomposes a time signal or a spatial image into the frequency domain, it is also known as the frequency domain representation of the original.

The idea of the fractional powers of the Fourier operator has been "discovered" several times in the literature. Initially, the idea appeared in the mathematical literature between First world War, 1914 to Second World War, 1945 (e.g., [3, 4]). A large number of publications relating to this idea appeared after Second World War, 1945. The fractional Fourier operator re-gains a momentum in 1980s with publications by Namias (e.g.[5]). Following Namias's contribution, a large number of papers appeared in the mathematical literature during 1990s to 2012s tying the concept of the fractional Fourier operators to mathematical analysis, Distribution theory, theoretical research and many other fields, Time-frequency

analysis as described in [6].

The fractional Fourier Transform (FRFT) is an elegant generalization of the ordinary Fourier transform [3–5, 7]. The fractional Fourier transform with a parameter α, has many applications in several areas including Communications, Optics, Quantum Physics and Singal processing. For more details of the fractional Fourier transform, see [8, 9]. The α-th order fractional Fourier transform represents the α-th power of the Fourier transform, when $\alpha = 90^0$, we obtain the Fourier transform, while $\alpha = 0^0$ for The fractional Fourier Transform (FRFT) is an elegant generalization of the ordinary Fourier transform [3–5, 7]. The fractional Fourier transform with a parameter α, has many applications in several areas including Communications, Optics, Quantum Physics and Singal processing. For more details of the fractional Fourier transform, see [8, 9]. The α-th order fractional Fourier transform represents the α-th power of the Fourier transform, when $\alpha = 90^0$, we obtain the Fourier transform, while for $\alpha = 0^0$, we obtain the signal itself. Any intermediate value of α (0^0 < α < 90^0) produces a signal representation that can be considered as a rotated timefrequency representation of the signal

[6]. The fractional Fourier transform with a parameter α, φ(x) $\in L_1(\mathbb{R})$ is defined as [8–12] α = 0^0, we obtain the signal itself. Any intermediate value of α (0^0 < α < 90^0) produces a signal representation that can be considered as a rotated time frequency representation of the signal [6].

The Fourier Transform of a complex-valued (Lebesgue) integrable function φ(t)$\epsilon L_1(\mathbb{R})$ on the real line is the complex valued function $\widehat{\phi}\,(\zeta)$ is defined by the integral as follows,

$$\widehat{\phi}\,(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta t}\,\phi(\mathrm{t})\mathrm{dt}$$

so that its inverse is given by

$$\phi(\mathrm{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta t}\widehat{\phi}\,(\zeta)\mathrm{d}\,\zeta < \infty$$

The fractional Fourier transform with a parameter α, φ(x) $\in L_1(\mathbb{R})$ is defined as [8–12]

$$(\mathfrak{F}_\alpha \phi)(\zeta) = \widehat{\phi_\alpha}\,(\zeta) = \int_{-\infty}^{\infty} K_\alpha\,(\eta,\zeta)\,\phi(\eta)d\eta$$

where the kernel $K_\alpha\,(\eta,\zeta)$ is given by

$$K_\alpha \;= \begin{cases} C_\alpha e^{\frac{i(\eta^2+\zeta^2)cot\alpha}{2} - \eta\zeta icsc\alpha,}, \alpha \neq n\pi, n \in \mathbb{Z} \\ \dfrac{1}{\sqrt{2\pi}} e^{-\eta\zeta i}, \alpha = \dfrac{1}{2\pi}, \end{cases}$$

and $C_\alpha = \sqrt{\frac{1-icot\alpha}{2\alpha}}$.

The inverse $of\;(\mathfrak{F}_\alpha \phi)(\zeta)$ is given by

$$\phi(\eta) = \int_{-\infty}^{\infty} \overline{K_\alpha\,(\eta,\zeta)}\,(\mathfrak{F}_\alpha \phi)(\zeta)d\zeta$$

and $\overline{K_\alpha\,(\eta,\zeta)} = K_{-\alpha}\,(\eta,\zeta)$.

The inverse of a FRFT with the parameter α is the FrFT with the parameter −α.

2 Future Scope in Technological Fields

Research scholars, Scientists, Faculties will solve numerous issues by implementing the FRFT. To

apply FRFT in hardware and software conditions, and To select the optimal order (OO) of FRFT. To answer these questions, we will give the recent/future advances for them.

2.1 Implementation of FRFT in Hardware

Hardware such as optical devices, digital signal processors (DSPs), field-programmable gate arrays (FP-GAs), and quantum gates will be used to implement FRFT. Tajahuerce (2000) [13] used a single imaging element (a blazed diffractive lens) to create an optical implementation of the FRFT with broadband illumination. Cai (2002) [14] proposed a simple two ordinary lens system to realise the FRFT of continuously variable order while keeping the scaling factors of both the input and output functions constant independent of the system's geometric parameters and FRFT orders. It could eliminate the need for and

inconvenience of using input masks of different scales or any other special devices, making it useful to those who wanted to observe the FRFT but lacked the necessary equipment. By using a realistic implementation of the DFRFT on a digital signal processor, Narayanan (2003) [15] demonstrated FRFT applications. Cai (2006) [16] proposed a lensless optical system for realising the coincidence FRFT effect. The lensless optical system conditions for implementing the coincidence FRFT with incoherent light and entangled photon pairs were discussed. The findings provided a novel scheme for FRFTs and thus suggested potential applications. Hahn (2006) [17] demonstrated and proposed an optical implementation of the iterative fractional Fourier transform (FRFT) algorithm. Sinha (2007) [18] presented a new configurable architecture for fast computation of the Centered Discrete Fractional Fourier Transform. Tao (2010) [19] presented an efficient implementation based on field-programmable gate array for performing fractional Fourier transform (FRFT) operations.

2.2 Implementation of FRFT in Software

Research scholars, Scientists, Faculties, Researchers are/will particularly interested in the numerical calculation of FRFT. Deng (1997) [20] proposed a fast implementation method for calculating fractional Fourier transforms based on the Chirp-Z transform. Ran (2000) [21] thought the classical discrete Fourier transform operator corresponded to a 4 x 4 cyclic matrix group, and the discrete fractional Fourier transform operator corresponded to a 4 x 4 generalized permutation matrix. Huang (2000) [22] provided a unified approach to improve the computation of the discrete fractional Fourier transform. They developed a general structure based on multi-rate filter to improve computation. Vundela (2013) [23] implemented the equivalent filter bank structures for the computation of the FRFT, in order to reduce the computation time. First two methods to compute the FRFT of any function was derived. Then the computation method was implemented using the filter bank approach.

3 Application

So far, FRFT has been used in a wide range of academic and industrial applications. Using the "Web of Science Core Collection" analytical tool, the following application categories are identified as the most popular: image processing, radiology, communication, encryption, optimal engineering, remote sensing, fractional calculus, fractional wavelet transform, pseudo-differential operator and pattern recognition.

3.1 Image Processing

Finally, FRFT will demonstrate successful image processing applications. Sun (2016) [24] used fractional Fourier entropy (FRFE) and multilayer perceptron to detect pathological brains. Sharma (2013) [25] proposed the use of DFRFT to replace conventional DFT in the Wiener and geometric mean filters. Sang (2013) [26] proposed a new FRFT-based image fusion method. All the source images were decomposed

by FRFT, and then the fractional orders were selected adaptively. It has well solved the poor adaptability of previous image fusion methods. Soni (2013) [26] illustrated the advantage of DFRFT as compared to other transforms for steganography in image processing. The simulation result showed same PSNR in both domain (time and frequency) but DFRFT gave an advantage of additional stego key i.e. order parameter of this transform. Liu (2013) [27] introduced FRFT to the three-dimensional model retrieval. Further, they proposed a three-dimensional model descriptor based on three dimensional FRFT. Guo (2011) [28] proposed a watermarking algorithm for optical images by the combination of random phase encoding and FRFT. Pan (2009) [29] developed a new accurate and adaptable scheme for calculating the polar FFT and the log-polar FFT, namely, Multilayer FRFT.

3.2 Radiology

FRFT has found several applications in radiology, which is a common medical specialty that generates images, On the basis of which physicians make diagnoses. Common imaging techniques include X-ray, computed tomography, ultrasound, positive emission tomography, and magnetic resonance imaging (MRI). Li (2016) [30] used FRFT to detect left-sided and right-sided hearing loss. Liu (2016) [31] used FRFT to make diagnosis for abnormal breasts. Ji (2015) [32] used FRFT and nonparallel SVM for classifying magnetic resonance brain images. Kumari (2013) [33] encrypted the DICOM images using a chained Hadamard transforms and number theoretic transforms. A secure fast 2D-DFRFT and a SPIHT

Algorithm with Huffman Encoder was utilized to compress the DICOM images. Zhang (2013) [34] focused on the medical image registration in the domain of FRFT. The Powell algorithm was implemented to optimize the parameters by evaluating the minimum MSE between the magnitudes of images. Two kinds of FRFT were used to construct the algorithms. Mustafi (2013) [35] proposed a new approach for medical image denoising. The proposed algorithm utilized the technique of both blind source separation and the FRFT. In the presence of quadratic field inhomogeneity, Irarrazaval (2011) [36] showed how fractional Fourier transform (FRFT) was used to reconstruct magnetic resonance (MR) images. Harput (2011) [37] proposed an ultrasound contact imaging method, with the aim of measuring the enamel thickness of the tooth. It was the first time that FRFT was employed in dental imaging.

3.3 Communication

FRFT will be applied by researchers/scientists to solve the problems in communication field. Lin (2011)[38] proposed a receiving system for impulse radio ultra-wideband communications. They divided received signals into the data and reference pulses using the energy concentration property of Chirp pulses in FRFD. White (2012) [39] presented analysis of detectors of LFM signals based on the FRFT. This allowed one to conduct a fair comparison of the performance of these methods with those based on the FT.

It was shown that the FRFT methods achieved superior performance if the sweep rate was sufficiently fast or the data length was sufficiently large. Wang (2013) [40] proposed a novel group FRFT based multiuser(MU) SISO Biorthogonal Frequency Division Multiplexing system, in which multiple independent data streams from co-channel mobile stations (MSs) can be transmitted in the same frequency and time slot. System simulations showed the essential advantages over conventional MU schemes.

3.4 Encryption

Some applications of FRFT will related to encryption in future. Zhou (2011) [41][136] proposed a novel image encryption (IE) algorithm method for chromatic images based on chaos and fractional Fourier transform (FRFT). Liu (2012) [42] offered an IE approach based on FRFT. They converted the data at local regions of complex function by FRFT. Bhatnagar (2014) [43] proposed a novel biometric inspired multimedia encryption technique. They proposed dual parameter FRFT (DP-FRFT) and used it in multimedia encryption.

3.5 Optical Engineering

FRFT will be introduced in the optical engineering. Gao (2010) [44] investigated the FRFT of the flattopped multi-Gaussian beam (FMGB) investigated for 3 types of fractional Fourier transform (FRFT) optical systems (OSs): Lohmann I and II, and quadratic graded-index systems.Tang (2012) [45] studiedthe propagation properties of confluent hypergeometric (HyG) beams propagating through FRFT OSs, based on FRFT in the cylindrical coordinate system.

3.6 Remote Sensing

An interesting application area of FRFT is remote sensing. Elgamel (2011) [46] presented a new signal processing algorithm, by an optimal fractional Fourier transform (FRFT) filtering. Wang (2012) [47] applied DFTFT so as to present a vibration estimation approach for synthetic aperture radar (SAR). Clemente (2012) [48] used FRFT to address a time variant problem for SAR processing.

3.7 Fractional Calculus

The idea of fractional can be generalized to other mathematical fields, such as fractional calculus, fractional partial differential equation (PDE), fractional variational iteration (FVI), fractional derivative operator (FDP), etc. Yang (2013) [49] investigated the transport equations in fractal porous media by using the fractional complex transform method. The local fractional linear and nonlinear transport equations with local fractional time and space fractional derivatives were obtained. The proposed models adequately described the fractal transport processes. Ray (2015) [50] considered the analytical solutions of fractional

PDEs with Riesz space fractional derivatives on a finite domain.

3.8 Fractional Wavelet Transform

Fractional wavelet transform (FRWT) was proposed to amend the limitations of fractional Fourier transform (FRFT) and wavelet transform (WT). Dinc (2010) [51] developed a new approach based on the combined use of the FRWT and the continuous wavelet transform (CWT) in order to quantify atorvastatin (ATO) and amlodipine (AML) in their mixtures without requiring a chemical pretreatment.

3.9 Pseudo-differential operator

Pseudo-differential operator is an extension of the differential operator. They are widely applied extensively in the theory of quantum field theory and PDEs. FRFT can be used to analyze the pseudo-differential operators. An integral representation of pseudo-differential operator and boundedness of the composition of operators and were obtained. An integral operator was defined and its boundedness property was studied.

3.10 Detection, Recognition, and Classification

FRFT was used by researchers to solve problems encountered in the disciplines of detection, recognition and classification. To simultaneously detect both the magnitude and direction of translation and tilt, Bhaduri (2010) [52] proposed a digital speckle photographic system, which implemented 2 simultaneous optical extended FRFTs.

4 Role of Fractional Fourier Transform in the Basic Sciences in Future Directions

Several topics are already very hot to use FRFT, such as SAR, encryption, etc. Nevertheless, several other research topics are not fully explored and have the potential to become hot in the future, since there are either few or increasing publications in those topics, which include but not limited to:

(i) Magnetic resonance imaging : Conventionally, the patient image is reconstructed by the DFT of scanned k-space [53–56]. Using FRFT may increase the reconstruction quality.

(ii) Fractional wavelet transform:- Wavelet transform achieves great success in various fields. FRWT was proposed by combining FRFT and WT. It had shown great success in early publications [51,57].However, its potential is not recognized fully.

(iii) Fruit Classification:- The fruit images are obtained in complicated conditions [58]: the positions and poses of cameras are different, the illumination conditions vary. Tuning the parameters of FRFT until the extracted time-frequency features are optimal for classification.

(iv) Image Processing :- There are a mass of papers discussing applications of traditional FT method to process images, and replacing FT by FRFT may increase the performance. Additional to it, the image denoising, image segmentation, image enhancement, and image reconstruction. All required more and more accurate signal processing that FRFT can provide. Cell image processing [59] is also a novel research method.

(v) Hybridization :- FRFT may be combined with automata, choice function [60], artificial intelligence and machine learning techniques, with the aim of developing a more efficient system.

(vi) Pathological brain detection: - The most important step is to extract efficient features from the brain images for detection [61]. The asymmetric structure of the brain can be extracted efficiently by FRFT.

(vii) Disease detection: - FRFT may help detect early symptoms of Alzheimers disease (AD) [62], multiple sclerosis (MS), hearing loss, abnormal breast, based on the unified-frequency domain.

(viii) Cloud Security: - FRFT may be used in cloud computer security [63] and information security. It can also help to protect the user’s privacy.

5 Conclusions

The FRFT method is a generalization of the classical FT method. It is a novel idea in time-frequency representation theory. This paper attempts to provide a thorough review of theoretical and technological points. FRFT applications in various fields are presented. We conclude that FRFT research is currently flourishing, and we anticipate. More results will be available in the coming years. Sustainable Future of Fractional Fourier Transform will essentially contribute to solve some challenges (Disease detection, Image Processing, Pathological brain detection etc.) of the people living on the earth.

References

[1] Joseph fourier, th´eorie analytique de la chaleur (1822). In Landmark Writings in Western Mathematics 1640-1940, pages 354–365. Elsevier, 2005

[2] James W Cooley and John W Tukey. An algorithm for the machine calculation of complex fourier series. Mathematics of computation, 19(90):297–301, 1965.

[3] Norbert Wiener. Hermitian polynomials and fourier analysis. Journal of Mathematics and Physics, 8(1-4):70–73, 1929.

[4] EU Condon. Immersion of the fourier transform in a continuous group of functional transformations. Proceedings of the National academy of Sciences of the United States of America, 23(3):158, 1937.

[5] Victor Namias. The fractional order fourier transform and its application to quantum mechanics. IMA Journal of Applied Mathematics, 25(3):241–265, 1980.

[6] Ervin Sejdi´c, Igor Djurovi´c, and Jin Jiang. Time–frequency feature representation using energy

concentration: An overview of recent advances. Digital signal processing, 19(1):153–183, 2009.

[7] H Ozaktas, Z Zalevsky, and MA Kutay. The fractional fourier transform with applications in optics and signal processing, jhon wiley & sons. New York, 2001.

[8] Luis B Almeida. The fractional fourier transform and time-frequency representations. IEEE Transactions on signal processing, 42(11):3084–3091, 1994.

[9] Ervin Sejdi´c, Igor Djurovi´c, and LJubiˇsa Stankovi´c. Fractional fourier transform as a signal processing tool: An overview of recent developments. Signal Processing, 91(6):1351–1369, 2011.

[10] Ahmed I Zayed. A convolution and product theorem for the fractional fourier transform. IEEE Signal processing letters, 5(4):101–103, 1998.

[11] Ahmed I. Zayed. Fractional fourier transform of generalized functions. Integral Transforms and Special Functions, 7(3-4):299–312, 1998.

[12] Gianfranco Cariolaro, Tomaso Erseghe, Peter Kraniauskas, and Nicola Laurenti. A unified framework for the fractional fourier transform. IEEE Transactions on Signal Processing, 46(12):3206–3219, 1998.

[13] Enrique Tajahuerce, Genaro Saavedra, Walter D Furlan, Enrique E Sicre, and Pedro Andres. White light optical implementation of the fractional fourier transform with adjustable order control. Applied optics, 39(2):238–245, 2000.

[14] LZ Cai and YQ Wang. Optical implementation of scale invariant fractional fourier transform of continuously variable orders with a two-lens system. Optics & Laser Technology, 34(3):249–252, 2002.

[15] V Ashok Narayanan and KMM Prabhu. The fractional fourier transform: theory, implementation and error analysis. Microprocessors and Microsystems, 27(10):511–521, 2003.

[16] Yangjian Cai and Fei Wang. Lensless optical implementation of the coincidence fractional fourier transform. Optics letters, 31(15):2278–2280, 2006.

[17] Joonku Hahn, Hwi Kim, and Byoungho Lee. Optical implementation of iterative fractional fourier transform algorithm. Optics Express, 14(23):11103–11112, 2006.

[18] Pavel Sinha, Saibal Sarkar, Amitabha Sinha, and Dhruba Basu. Architecture of a configurable centered discrete fractional fourier transform processor. In 2007 50th Midwest Symposium on Circuits and Systems, pages 329–332. IEEE, 2007

[19] Ran Tao, Guangping Liang, and Xing-Hao Zhao. An efficient fpga-based implementation of fractional fourier transform algorithm. Journal of Signal Processing Systems, 60:47–58, 2010.

[20] Xuegong Deng, Yongping Li, Dianyuan Fan, and Yue Qiu. A fast algorithm for fractional fourier transforms. Optics Communications, 138(4-6):270–274, 1997.

[21] QW Ran, YJ Feng, JZ Wang, and QT Wu. The discrete fractional fourier transform and its simulation. Chinese Journal of Electronics, 9(1):70–75, 2000.

[22] Der-Feng Huang and Bor-Sen Chen. A multi-input–multi-output system approach for the computation of discrete fractional fourier transform. Signal Processing, 80(8):1501–1513, 2000.

[23] Ujwala Vundela, T Sunil Kumar, and R Suresh. Computation of fractional fourier transform using filter bank approach and its application.In 2013 International Conference on Computer Communication and Informatics, pages 1–5. IEEE, 2013.

[24] Yudong Zhang, Yi Sun, Preetha Phillips, Ge Liu, Xingxing Zhou, and Shuihua Wang. A multilayer perceptron based smart pathological brain detection system by fractional fourier entropy. Journal of medical systems, 40:1–11, 2016.

[25] KK Sharma and Priya Mittal. Investigations on use of fractional fourier transform for image restoration in the wiener and geometric mean filters. In International Conference on Communication and Electronics System Design, volume 8760, pages 424–429. SPIE, 2013.

[26] Yudong Zhang, Shuihua Wang, Jian-Fei Yang, Zheng Zhang, Preetha Phillips, Ping Sun, and Jie Yan. A comprehensive survey on fractional fourier

transform. Fundamenta Informaticae, 151(1-4):1–48, 2017.

[27] Liu YuJie, Bao Feng, Li ZongMin, and Li Hua. 3d model retrieval based on 3d fractional fourier transform. The International Arab Journal of Information Technology, 10(5), 2013.

[28] Qing Guo, Zhengjun Liu, and Shutian Liucora. Image watermarking algorithm based on fractional fourier transform and random phase encoding. Optics Communications, 284(16-17):3918–3923, 2011.

[29] Wei Pan, Kaihuai Qin, and Yao Chen. An adaptable-multilayer fractional fourier transform approach for image registration. IEEE Transactions on Pattern Analysis and Machine Intelligence, 31(3):400–414, 2008.

[30] Shuihua Wang, Ming Yang, Yin Zhang, Jianwu Li, Ling Zou, Siyuan Lu, Bin Liu, Jiquan Yang, and Yudong Zhang. Detection of left-sided and right-sided hearing loss via fractional fourier transform. Entropy, 18(5):194, 2016.

[31] Yu-Dong Zhang, Shui-Hua Wang, Ge Liu, and Jiquan Yang. Computer-aided diagnosis of abnormal breasts in mammogram images by weighted-type fractional fourier transform. Advances in Mechanical Engineering, 8(2):1687814016634243, 2016.

[32] YD Zhang, GL Ji, SH Wang, JQ Yang, ZC Dong, and P Phillips. Bbo improves tumor detection in mri scanning. In Basic & Clinical Pharmacology & Toxicology, volume 117, pages 19–19. WILEY-BLACKWELL 111 RIVER ST, HOBOKEN 07030-5774, NJ USA, 2015.

[33] P Vasanthi Kumari and K Thanushkodi. A secure fast 2ddiscrete fractional fourier transform based medical image compression using hybrid encoding technique. In 2013 international conference on current trends in engineering and technology (ICCTET), pages 1–7. IEEE, 2013.

[34] XiuJie Zhang, Yi Shen, ShiYong Li, and HaiYing Zhang. Medical image registration in fractional fourier transform domain. Optik, 124(12):1239–1242, 2013.

[35] A Mustafi and SK Ghorai. A novel blind source separation technique using fractional fourier transform for denoising medical images. Optik, 124(3):265–271, 2013.

[36] Pablo Irarrazaval, Carlos Lizama, Vicente Parot, Carlos Sing-Long, and Cristian Tejos. The fractional fourier transform and quadratic field magnetic resonance imaging. Computers & Mathematics with Applications, 62(3):1576–1590, 2011.

[37] Sevan Harput, Tony Evans, Nigel Bubb, and Steven Freear. Diagnostic ultrasound tooth imaging using fractional fourier transform. IEEE transactions on ultrasonics, ferroelectrics, and frequency control, 58(10):2096–2106, 2011.

[38] Di Lin, XuanLi Wu, and XueJun Sha. Fractional fourier transform based transmitted reference scheme for uwb communications. Science China Information Sciences, 54:1712–1722, 2011.

[39] Paul R White and J Locke. Performance of methods based on the fractional fourier transform for the detection of linear frequency modulated signals. IET signal processing, 6(5):478–483, 2012.

[40] Huiqi Wang. A novel multiuser siso-bofdm systems with group fractional fourier transforms scheme. Wireless Personal Communications, 69:735–743, 2013.

[41] Nanrun Zhou, Yixian Wang, Lihua Gong, Hong He, and Jianhua Wu. Novel single-channel color image encryption algorithm based on chaos and fractional fourier transform. Optics Communications, 284(12):2789–2796, 2011.

[42] Zhengjun Liu, Lie Xu, Jingmin Dai, and Shutian Liu. Image encryption by using local random phase encoding in fractional fourier transform domains. Optik, 123(5):428–432, 2012.

[43] Gaurav Bhatnagar and QM Jonathan Wu. Biometric inspired multimedia encryption based on dual parameter fractional fourier transform. IEEE transactions on systems, man, and cybernetics: systems, 44(9):1234–1247, 2014.

[44] Yan-Qi Gao, Bao-Qiang Zhu, Dai-Zhong Liu, and Zun-Qi Lin. Fractional fourier transform of flattopped multi-gaussian beams. JOSA A, 27(2):358–365, 2010.

[45] Bin Tang, Chun Jiang, and Haibin Zhu. Fractional fourier transform for confluent hypergeometric beams. Physics Letters A, 376(38-39):2627–2631,

2012.

[46] Sherif A Elgamel and John Soraghan. Enhanced monopulse tracking radar using optimum fractional fourier transform. IET radar, sonar & navigation, 5(1):74–82, 2011.

[47] Qi Wang, Matthew Pepin, Ryan J Beach, Ralf Dunkel, Tom Atwood, Balu Santhanam, Walter Gerstle, Armin W Doerry, and Majeed M Hayat. Sar-based vibration estimation using the discrete fractional fourier transform. IEEE Transactions on Geoscience and Remote Sensing, 50(10):4145–4156, 2012.

[48] Carmine Clemente and John J Soraghan. Range doppler and chirp scaling processing of synthetic aperture radar data using the fractional fourier transform. IET signal processing, 6(5):503–510, 2012.

[49] Xiao-Jun Yang, Dumitru Baleanu, and Ji-Huan He. Transport equations in fractal porous media within fractional complex transform method. Proceedings of the Romanian Academy A, 14(4):287–292, 2013.

[50] S Saha Ray and S Sahoo. Analytical approximate solutions of riesz fractional diffusion equation and riesz fractional advection–dispersion equation involving nonlocal space fractional derivatives.

[51] Mathematical Methods in the Applied Sciences, 38(13):2840–2849, 2015.

Erdal Dinc¸ and Dumitru Baleanu. Fractional wavelet transform for the quantitative spectral resolution of the composite signals of the active compounds in a two-component mixture. Computers &

Mathematics with Applications, 59(5):1701–1708, 2010.

[52] Basanta Bhaduri, CJ Tay, C Quan, and Colin JR Sheppard. Motion detection using extended fractional fourier transform and digital speckle photography. Optics Express, 18(11):11396–11405, 2010.

[53] Yudong Zhang, Shuihua Wang, Genlin Ji, and Zhengchao Dong. Exponential wavelet iterative shrink-age thresholding algorithm with random shift for compressed sensing magnetic resonance imaging. IEEJ Transactions on Electrical and Electronic Engineering, 10(1):116–117, 2015.

[54] Zhengchao Dong, Yudong Zhang, Feng Liu, Yunsuo Duan, Alayar Kangarlu, and Bradley S Peterson. Improving the spectral resolution and spectral fitting of 1h mrsi data from human calf muscle by the spread technique. NMR in Biomedicine, 27(11):1325–1332, 2014.

[55] Suzanne Goh, Zhengchao Dong, Yudong Zhang, Salvatore DiMauro, and Bradley S Peterson. Mito-chondrial dysfunction as a neurobiological subtype of autism spectrum disorder: evidence from brain imaging. JAMA psychiatry, 71(6):665–671, 2014.

[56] Yudong Zhang, Zhengchao Dong, Preetha Phillips, Shuihua Wang, Genlin Ji, and Jiquan Yang. Ex-ponential wavelet iterative shrinkage thresholding algorithm for compressed sensing magnetic resonance imaging. Information Sciences, 322:115–132, 2015.

[57] Mustafa C¸ elebier, Sacide Altın¨oz, and Erdal Dinc¸. Fractional wavelet transform and chemometric calibrations for the simultaneous determination of amlodipine and valsartan in their complex mixture. New Trends in Nanotechnology and Fractional Calculus Applications, pages 333–340, 2010.

[58] Shuihua Wang, Yudong Zhang, Genlin Ji, Jiquan Yang, Jianguo Wu, and Ling Wei. Fruit classification by wavelet-entropy and feedforward neural network trained by fitness-scaled chaotic abc and biogeography-based optimization. Entropy, 17(8):5711–5728, 2015.

[59] Yudong Zhang, Bo Peng, Shuihua Wang, Yu-Xiang Liang, Jiquan Yang, Kwok-Fai So, and Ti-Fei Yuan. Image processing methods to elucidate spatial characteristics of retinal microglia after optic nerve transection. Scientific Reports, 6(1):1–7, 2016.

[60] Arnaud Carayol, Christof L¨oding, Damian Niwinski, and Igor Walukiewicz. Choice functions and well-orderings over the infinite binary tree. Open Mathematics, 8(4):662–682, 2010.

[61] Yu-Dong Zhang, Shuihua Wang, and Zhengchao Dong. Classification of alzheimer disease based on structural magnetic resonance imaging by kernel support vector machine decision tree. Progress InElectromagnetics Research, 144:171–184, 2014.

[62] Simon F Eskildsen, Pierrick Coup´e, Vladimir S Fonov, Jens C Pruessner, D Louis Collins,

Alzheimer's Disease Neuroimaging Initiative, et al. Structural imaging biomarkers of alzheimer's disease: predicting disease progression. Neurobiology of aging, 36:S23–S31, 2015.

[63] Yi Sun, Qiaoyan Wen, Yudong Zhang, and Wenmin Li. Privacy-preserving self-helped medical diagnosis scheme based on secure two-party computation in wireless sensor networks. Computational and mathematical methods in medicine, 2014, 2014.