Dominating Sequential Functions in Context of Nonelementary and Hypergeometric Functions

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Abstract

Function plays an important role in mathematics and indefinite integration (i.e. antiderivative) provides the opportunity to create and propound new functions. Special functions like hypergeometric function, error function, exponential function etc. are useful as a tool of expressing many elementary and nonelementary functions as well as nonelementary integrals. In this paper we have studied the relations between dominating sequential and hypergeometric functions in context of nonelementary functions. The dominating sequential functions contain the extended dominating sequential trigonometric, dominating sequential hyperbolic, dominating sequential exponential and dominating sequential logarithmic functions. The dominating sequential functions have been expressed in terms of hypergeometric functions. A relation between dominating sequential and nonelementary functions has been discussed and found that no general relation exists between them except for some particular examples in terms of antiderivative. Also there doesn't exist any direct or indirect relation between sequential and nonelementary function. The paper ends with a short note on limitations of the work and the future scope of research based on this paper.

Keywords: Function, elementary function, nonelementary function, hypergeometric function, dominating function, dominating sequential function, etc.

1. Introduction

The function contributed a lot in the development of mathematics and its allied sciences with their applications. Being a relation between a set of inputs and a set of outputs, it enters whenever independent and dependent variables are connected with a definite relationship (Function (mathematics) - Wikipedia contributors, 2025; Chaudhary & Yadav, 2024). The concept of function was elaborated with the calculus at the end of 17th century. From the start of 18th to the end of 19th centuries, it was formalized to express mathematical formulae to show the exact nature of relationship. Leibnitz (1646-1716) was the first mathematician, who used the term function (Courant et al., 2021; Function (mathematics) - Wikipedia contributors, 2025; Hannah, 2013; Katz, 2019; Vygodsky, 1987).

Functions can be grouped in different ways but for the present work, we will divide it in elementary and nonelementary functions. A function f is called an elementary function if it can be expressed in the form y = f(x), where f(x) represents an expression formed by the combination of a finite number of terms using the exponents of x, constant (arbitrary or fixed), trigonometric functions, hyperbolic functions, exponential functions, logarithmic functions, their inverses, etc. together through addition, subtraction, multiplication, division, powers and compositions. The function which cannot be expressed in terms of elementary function form is known as nonelementary functions. In our study nonelementary integrals are also treated as nonelementary functions. The elementary function can always be expressed in closed form (Cherry, 1985, 1986; Closed - form expression - Wikipedia contributors, 2025; Elementary function - Wikipedia contributors, 2025; Function (mathematics) - Wikipedia contributors, 2025; Chaudhary & Yadav, 2024; Gale et al., 1923; Kasper, 1980; Marchisotto et al., 1994; Muller, 2006; Risch, 1969, 1970, 2022; Ritt, 2022; Rosenlicht, 1972).

Beside these two functions, there is one special function known as hypergeometric function, which is in general represented by a series called hypergeometric series and thereafter denoted by three major notations known as hypergeometric functions with different names. Out of three the general hypergeometric function is denoted and defined by

$$mFn(\propto_1,\propto_2,\dots,\propto_m;\beta_1,\beta_2,\dots,\beta_n;x)$$
$$=\sum_{r=0}^{\infty}\frac{(\propto_1)_r.(\propto_2)_r,\dots,(\propto_m)_r}{(\beta_1)_r,(\beta_2)_r,\dots,(\beta_n)_r}\frac{x^r}{r!}$$
$$=1+\frac{\propto_1\propto_2\dots\propto_m}{\beta_1\beta_2\dots\beta_n}\frac{x^1}{1!}$$
$$+\frac{\propto_1(\propto_1+1)\propto_2(\propto_2+1)\dots\propto_m(\propto_m+1)}{\beta_1(\beta_1+1)\beta_2(\beta_2+1)\dots\beta_n(\beta_n+1)}\frac{x^2}{2!}+\cdots$$

where

$$(\alpha)_r=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+r-1)$$

and $(\alpha)_0=1$ known as shifted factorial. The notation

$$mFn(\propto_1,\propto_2,\dots,\propto_m;\beta_1,\beta_2,\dots,\beta_n;x)$$

is called hypergeometric function and the series on the right hand side is called the hypergeometric series (Bailey, 1964; Chaundy, 1943; Chaudhary & Yadav, 2024; Du et al., 2002; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2025).

The second one confluent hypergeometric function is obtained from the above by putting m = n = 1 as

$$1F1(\propto,\beta,x)=\sum_{r=0}^{\infty}\frac{(\propto)_r}{(\beta)_r}\frac{x^r}{r!}$$
$$=1+\frac{\propto}{\beta}\frac{x}{1!}+\frac{\propto(\propto+1)}{\beta(\beta+1)}\frac{x^2}{2!}+\cdots$$

(Chaudhary & Yadav, 2024; Confluent hypergeometric function - Wikipedia contributors, 2025; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2025).

The third one is Gauss hypergeometric function, which is obtained by putting m = 2, n = 1 in general hypergeometric function as

$$F(\propto,\beta;\gamma;x)=2F1(\propto,\beta;\gamma;x)=\sum_{r=0}^{\infty}\frac{(\propto)_r.(\beta)_r}{(\gamma)_r}\frac{x^r}{r!}$$
$$=1+\sum_{r=1}^{\infty}\frac{(\propto)_r.(\beta)_r}{(\gamma)_r}\frac{x^r}{r!}=1+\frac{\propto\beta}{\gamma}\frac{x}{1!}$$
$$+\frac{\propto(\propto+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\cdots$$

where x, ∝, β, γ may be real or complex, $|x|<1$ and γ is a non-negative integer (Chaudhary & Yadav, 2024; Hannah, 2013; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2025). In our study we will use the notations $F(\propto,\beta;\gamma;x)$, ${}_2F_1(\propto,\beta;\gamma;x)$ and $2F1(\propto,\beta;\gamma;x)$ for it.

We know that integration plays an important role to create new functions. Based on it Yadav & Sen (2009, 2012, 2017, 2025) propounded dominating function, sequential function and dominating sequential functions to find the suitable function to express nonelementary integrals in compact form. The study of domination is the fastest areas in mathematics and allied sciences. It was given a formal definition by Berge in 1958 and Ore in 1962. Although the idea of dominating sets has its origin in the game of chess, in which we study sets of chess pieces which cover various opposing pieces or various squares of the chess board. One such popular problem is known as 'queen's domination problem'. John Van Neumann also considered the term domination in diagraphs while solving problems in game theory. The introduction of different dominating functions by Yadav & Sen is also a milestone (Dominating set - Wikipedia contributors, 2025; Yadav et al., 2009; Yadav & Sen, 2025; Yadav & Yadav, 2025).

Yadav et al. (2009, 2025) proposed the dominating function represented by a two sided infinite series as

$$f(x)=\sum_{n=1}^{\infty}\frac{B_n}{(ax+b)^n}+C_0+\sum_{n=1}^{\infty}A_n(ax+b)^n$$

where A_n,B_n,C_0 are all real constants for all real n. They called a function f(x) dominatable or d-able if it can be expressed in the above form and proved that all most all elementary functions are d-able functions.

Yadav & Sen (2012, 2017) propounded dominating sequential functions for all types of elementary functions like of trigonometric, hyperbolic, exponential and logarithmic functions, which dominate the classical elementary trigonometric, hyperbolic, exponential and logarithmic functions in the sense that for particular values of some arbitrary constants, the dominating sequential functions reduces to the classical elementary functions.

Recently Yadav & Yadav (2024) have reviewed the nonelementary functions (and integrals) in context of hypergeometric functions. Chaudhary & Yadav (2024) studied the hypergeometric functions in context of elementary and nonelementary integrals (and functions). They discussed eleven propositions based on five questions on elementary functions, nonelementary functions, hypergeometric functions and nonelementary integrals. The present paper is an extension of these works for dominating sequential functions in context of hypergeometric functions.

2. Preliminary Ideas

For establishing the relations between dominating sequential and hypergeometric functions, let us have some basic ideas of mathematical terminology:

Elementary Functions: The functions like algebraic, trigonometric, hyperbolic, exponential and logarithmic functions have been treated as elementary functions, because all these can be expressed in a closed form expression (Closed form - Wikipedia contributors, 2025; Function (mathematics) - Wikipedia contributors, 2025; Marchisotto et al., 1994; Risch, 1969, 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav, 2015; Yadav & Sen, 2025).

Dominating Elementary Functions: We have called dominating trigonometric functions, dominating hyperbolic functions, dominating exponential functions and dominating logarithmic functions as dominating elementary functions (Yadav et al., 2009, 2012, 2025).

Sequential Function: Yadav & Sen (2012) have propounded sequential functions for trigonometric, hyperbolic, exponential and logarithmic functions based on the inner product of two n-vectors of a general sequence and elementary functions. Thereafter they extended it for different dominating functions and called them dominating sequential functions. Sequential functions were obtained by expressing the sequence and function in series and then taking their inner product as follows:

Let us consider a sequence

$$\left\{\frac{a_n}{b_n}\right\}_{n\geq 1} = \left\{\frac{a_{n+1}}{b_{n+1}}\right\}_{n\geq 0} = \left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots, \frac{a_{n+1}}{b_{n+1}}, \dots\right\}$$

and the series expansion of sinx as

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} - \dots$$
$$= \left\{x, -\frac{x^3}{3!}, \frac{x^5}{5!}, \dots, \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \dots\right\}$$

Taking the inner product of the above two infinite sequences, we get

$$\left\{\frac{a_{n+1}}{b_{n+1}}\right\}_{n\geq 0} . sinx$$
$$= \left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots, \frac{a_{n+1}}{b_{n+1}}, \dots\right\}. \left\{x, -\frac{x^3}{3!}, \frac{x^5}{5!}, \dots, \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \dots\right\}$$

$$= \left\{\frac{a_1}{b_1}x, -\frac{a_2}{b_2}\frac{x^3}{3!}, \frac{a_3}{b_3}\frac{x^5}{5!}, \dots, \frac{a_{n+1}}{b_{n+1}}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, \dots\right\}$$
$$= \frac{a_1}{b_1}x - \frac{a_2}{b_2}\frac{x^3}{3!} + \frac{a_3}{b_3}\frac{x^5}{5!} - \dots + \frac{a_{n+1}}{b_{n+1}}\frac{(-1)^n x^{2n+1}}{(2n+1)!} - \dots$$
$$= sin\left\{\frac{a_{n+1}}{b_{n+1}}\right\}x$$

i.e.

$$sin\left\{\frac{a_{n+1}}{b_{n+1}}\right\}x = \sum_{n=0}^{\infty}\left\{\frac{a_{n+1}}{b_{n+1}}\right\}.\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and

$$sin\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

We adjust the sequence according as the lower limit of n in the summation. Similarly other sequential trigonometric functions have been denoted and defined by Yadav & Sen (2012) as

$$cos\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!}$$

$$tan\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=0}^{\infty}\left\{\frac{a_{n+1}}{b_{n+1}}\right\}.\frac{(-1)^n 2^{2n+2}(2^{2n+2}-1)B_{2n+2}x^{2n+1}}{(2n+2)!}$$

$$cot\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}2^{2n-2}B_{2n-2}x^{2n-3}}{(2n-2)!}$$

$$cosec\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^n 2(2^{2n-3}-1)B_{2n-2}x^{2n-3}}{(2n-2)!}$$

$$sec\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}E_{2n-2}x^{2n-2}}{(2n-2)!}$$

where B, E are called Bernoulli and Euler numbers. When we apply these concepts on dominating trigonometric functions, we get dominating sequential trigonometric functions (Yadav & Sen, 2012, 2017, 2025) as follows:

$$dsin\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}x^{(2n-1)r-m}}{(2n-1)!}$$

$$dcos\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}x^{2nr-m-2r}}{(2n-2)!}$$

$$dtan\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{(2n-1)r-m}}{(2n)!}$$

$$dcot\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}2^{2n-2}B_{2n-2}x^{(2n-3)r-m}}{(2n-2)!}$$

$$dcosec\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^n 2(2^{2n-3}-1)B_{2n-2}x^{(2n-3)r-m}}{(2n-2)!}$$

$$dsec\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}E_{2n-2}x^{2nr-m-2r}}{(2n-2)!}$$

where m, r are constants and can take any arbitrary constant on demand. Following the same procedures, Yadav & Sen (2012) propounded the sequential hyperbolic functions as

$$sinh\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{x^{2n-1}}{(2n-1)!}$$

$$cosh\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{x^{2n-2}}{(2n-2)!}$$

$$tanh\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)|B_{2n}|x^{2n-1}}{(2n)!}$$

$$coth\left\{\frac{a_n}{b_n}\right\}x = \frac{a_1}{b_1}\frac{1}{x} + \sum_{n=1}^{\infty}\left\{\frac{a_{n+1}}{b_{n+1}}\right\}.\frac{2^{2n}B_{2n}x^{2n-1}}{(2n)!}$$

$$cosech\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}2|(2^{2n-3}-1)||B_{2n-2}|x^{2n-3}}{(2n-2)!}$$

$$sech\left\{\frac{a_n}{b_n}\right\}x = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}|E_{2n-2}|x^{2n-2}}{(2n-2)!}$$

and the dominating sequential hyperbolic functions as

$$dsinh\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{x^{(2n-1)r-m}}{(2n-1)!}$$

$$dcosh\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{x^{2nr-m-2r}}{(2n-2)!}$$

$$dtanh\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)|B_{2n}|x^{(2n-1)r-m}}{(2n)!}$$

$$dcoth\left\{\frac{a_n}{b_n}\right\}x_m^r = \frac{a_1}{b_1}\frac{1}{x^{m+r}} + \sum_{n=1}^{\infty}\left\{\frac{a_{n+1}}{b_{n+1}}\right\}.\frac{2^{2n}B_{2n}x^{(2n-1)r-m}}{(2n)!}$$

$$dcosech\left\{\frac{a_n}{b_n}\right\}x_m^r = \sum_{n=1}^{\infty}\left\{\frac{a_n}{b_n}\right\}.\frac{(-1)^{n-1}2|(2^{2n-3}-1)||B_{2n-2}|x^{(2n-3)r-m}}{(2n-2)!}$$

$$dsech\left\{\frac{a_n}{b_n}\right\}x_m^r$$

$$= \sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . \frac{(-1)^{n-1}|E_{2n-2}|x^{2nr-m-2r}}{(2n-2)!}$$

Similarly Yadav & Sen (2012) propounded sequential exponential, dominating sequential exponential, sequential logarithmic and dominating sequential logarithmic functions as

$$e^{\left\{\frac{a_n}{b_n}\right\}x} = \sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . \frac{x^{n-1}}{(n-1)!}$$

$$de^{\left\{\frac{a_n}{b_n}\right\}x_m^r} = \sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . \frac{x^{nr-r-m}}{(n-1)!}$$

$$ln \left\{\frac{a_n}{b_n}\right\} (1+x) = \sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . (-1)^{n+1} \frac{x^n}{n}$$

$$ln \left\{\frac{a_n}{b_n}\right\} (1-x) = -\sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . \frac{x^n}{n}$$

$$dln \left\{\frac{a_n}{b_n}\right\} (1+x_m^r) = \sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . (-1)^{n+1} \frac{x^{nr-m}}{n}$$

$$dln \left\{\frac{a_n}{b_n}\right\} (1-x_m^r) = -\sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . \frac{x^{nr-m}}{n}$$

In the present paper we attempted to correlate the above functions with hypergeometric functions and nonelementary functions.

3. Discussion

As far as the relations between elementary, nonelementary, dominating elementary, sequential, dominating sequential and hypergeometric functions are concerned, we can discuss them one by one. We know that “every elementary function is expressible in terms of hypergeometric functions” either directly or indirectly (Chaudhary & Yadav, 2024). Recently Yadav & Yadav (2025) have propounded that “dominating elementary functions can also be expressed in terms of hypergeometric functions” either directly or indirectly. We also know that elementary and nonelementary functions are interrelated by antiderivative of elementary functions and Chaudhary & Yadav (2024) have stated that “all nonelementary functions are not necessarily hypergeometric functions”.

As far as the sequential function is concerned, we see that

$$sin \left\{\frac{a_n}{b_n}\right\} x = \left\{\frac{a_n}{b_n}\right\} . sinx$$

$$= \sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$$

Here $\frac{x^{2n-1}}{(2n-1)!}$ can be adjusted with the term $\frac{x^p}{p!}$ of the hypergeometric series but the term of the sequence $\left\{\frac{a_n}{b_n}\right\}$ cannot be adjusted with the term of the coefficient $\frac{(\propto)_r.(\beta)_r}{(\gamma)_r}$ of hypergeometric series because the sequence $\left\{\frac{a_n}{b_n}\right\}$ contains only one term in each expansion of the sequence for all n, whereas in the coefficient containing three (may be less or more) rising factorials, the number of terms increases in each individual terms and it is obtained by the product of all previous terms for each n. Thus $sin \left\{\frac{a_n}{b_n}\right\} x$ cannot be expressed in terms of hypergeometric series i.e. in terms of hypergeometric functions. Similar logic would be applicable for all the sequential functions like

$$cos \left\{\frac{a_n}{b_n}\right\} x, tan \left\{\frac{a_n}{b_n}\right\} x, cot \left\{\frac{a_n}{b_n}\right\} x, cosec \left\{\frac{a_n}{b_n}\right\} x,$$

$$sec \left\{\frac{a_n}{b_n}\right\} x, sinh \left\{\frac{a_n}{b_n}\right\} x, cosh \left\{\frac{a_n}{b_n}\right\} x,$$

$$tanh \left\{\frac{a_n}{b_n}\right\} x, coth \left\{\frac{a_n}{b_n}\right\} x, cosech \left\{\frac{a_n}{b_n}\right\} x,$$

$$sech \left\{\frac{a_n}{b_n}\right\} x, e^{\left\{\frac{a_n}{b_n}\right\}x}, ln \left\{\frac{a_n}{b_n}\right\} (1+x),$$

$$ln \left\{\frac{a_n}{b_n}\right\} (1-x), etc.$$

Thus we can conclude that sequential trigonometric, sequential hyperbolic, sequential exponential and sequential logarithmic functions cannot be expressed in terms of hypergeometric functions i.e. “sequential functions are not expressible in terms of hypergeometric functions”.

Consequently dominating sequential trigonometric functions cannot be expressed in terms of hypergeometric functions because we can see that

$$dsin \left\{\frac{a_n}{b_n}\right\} x_m^r = \left\{\frac{a_n}{b_n}\right\} . sinx_m^r$$

$$= \sum_{n=1}^{\infty} \left\{\frac{a_n}{b_n}\right\} . \frac{(-1)^{n-1}x^{(2n-1)r-m}}{(2n-1)!}$$

Here $\frac{x^{(2n-1)r-m}}{(2n-1)!}$ can be adjusted with the term $\frac{x^q}{q!}$ of the hypergeometric series but the term of the sequence $\left\{\frac{a_n}{b_n}\right\}$ cannot be adjusted with the term of the coefficient $\frac{(\propto)_r.(\beta)_r}{(\gamma)_r}$ of hypergeometric series according to the reasons given above. Thus $dsin\left\{\frac{a_n}{b_n}\right\}x_m^r$ cannot be expressed in terms of hypergeometric series i.e. in terms of hypergeometric functions. Similar logic can be applied for all the dominating sequential functions like

$$dcos\left\{\frac{a_n}{b_n}\right\}x_m^r, dtan\left\{\frac{a_n}{b_n}\right\}x_m^r, dcot\left\{\frac{a_n}{b_n}\right\}x_m^r,$$

$$dcosec\left\{\frac{a_n}{b_n}\right\}x_m^r, dsec\left\{\frac{a_n}{b_n}\right\}x_m^r,$$

$$dsinh\left\{\frac{a_n}{b_n}\right\}x_m^r, dcosh\left\{\frac{a_n}{b_n}\right\}x_m^r, dtanh\left\{\frac{a_n}{b_n}\right\}x_m^r,$$

$$dcoth\left\{\frac{a_n}{b_n}\right\}x_m^r, dcosech\left\{\frac{a_n}{b_n}\right\}x_m^r, dsech\left\{\frac{a_n}{b_n}\right\}x_m^r,$$

$$de^{\left\{\frac{a_n}{b_n}\right\}x_m^r}, dln\left\{\frac{a_n}{b_n}\right\}(1+x_m^r),$$

$$dln\left\{\frac{a_n}{b_n}\right\}(1-x_m^r), etc.$$

From above discussion we conclude that dominating sequential trigonometric, dominating sequential hyperbolic, dominating sequential exponential and dominating sequential logarithmic functions are not expressible in terms of hypergeometric functions i.e. "dominating sequential functions are not expressible in terms of hypergeometric functions".

Thus we conclude that dominating sequential functions cannot be expressed in terms of hypergeometric functions, It is also clear that dominating sequential functions are elementary functions in the sense that they can be written in closed form using finite number of elementary functions. As far as the relation between nonelementary functions and dominating sequential functions is concerned, Yadav & Sen (2012, 2025) have expressed some nonelementary functions (integrals) in terms of dominating sequential function as

$$\int \frac{sinx}{x}dx = sin\left\{\frac{1}{(2n+1)}\right\}x + K, n \geq 0$$

$$\int \frac{cosx}{x}dx = lnx + \left[dcos\left\{\frac{1}{2n}\right\}x_0^1\right] + K, n > 0$$

where K is an arbitrary constant of integration. In above two integrals, both are nonelementary but their integrals can be expressed in terms of sequential and dominating sequential functions. But it is not always true for other nonelementary functions.

4. Conclusion

From the discussion we can conclude that although some nonelementary integrals (functions) are expressible in terms of sequential and dominating sequential function but not all nonelementary functions are expressible in terms of sequential and dominating sequential functions. Similarly we found that the sequential and dominating sequential functions are not expressible in terms of hypergeometric functions. Thus no general relation is found between dominating sequential functions and nonelementary functions.

5. Limitations and Future Scope of Research

There is a scope of advancement of new functions for dominating sequential functions in notations as well as a deep study on nonelementary functions (integrals) in context of dominating sequential functions can be done. Because dominating sequential functions have been studied in context of nonelementary functions for particular examples only.

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