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A Study on the Fourier Transforms with Their Utilizations

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Abstract: In applied sciences, Fourier transforms are crucial for comprehending a range of physical processes. The application of Fourier Transforms is particularly beneficial in the computational sciences. In this work, we discuss continuous Fourier Transforms, Discrete Fourier Transforms, and their utilizations in basic science and engineering i.e. in Communications, Optics, Quantum Physics and Singal processing, noise reduction, image compression, Biomedical Engineering etc. The uses of the one-dimensional and two-dimensional Fourier transforms in image processing and signal noise filtering have been discussed here.

Keywords: Fourier Transform, Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT), Image Processing, Biomedical Engineering.

1. Introduction

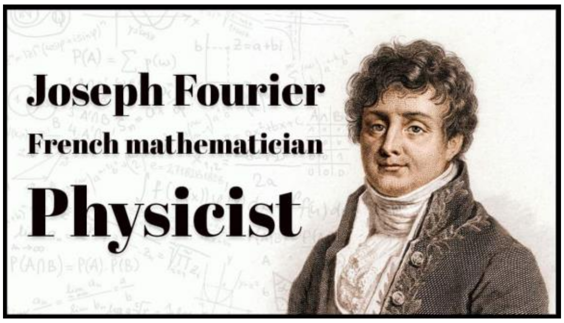

Figure 1: Jean-Baptiste Joseph Fourier.

In honour of Jean-Baptiste Joseph Fourier (1768–1830), who made significant contributions to the study of trigonometric series, the Fourier series was named. The Fourier Series is named for Jean

Baptiste Joseph Fourier, following preliminary study by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli. In the beginning, Euler gave the formula for the coefficients of the Fourier Series.

Based on Euler's study, Clairaut penned what is today regarded as the first formula for the Discrete Fourier transform (DFT) in 1754. In 1805, Carl Friedrich Gauss (1777–1855) presented a DFT formula that does not need to interpolate only using odd or even periodic functions. The Fourier Transform [1] got its name from a paper on heat flow written by Joseph Fourier (1768–1830) in 1822.

Jean-Baptiste Joseph Fourier studied

$$\varphi(t) = a_0 \cos\frac{\pi t}{2} + a_1 \cos 3\frac{\pi t}{2} + a_2 \cos 5\frac{\pi t}{2} + \dots\dots.$$

where $a_n = \int_{-1}^{1} \varphi(t) \cos(2n+1)\frac{\pi t}{2} dt.$

Double Fourier Series: The principle of Fourier series expansion for a univariate function t may be extended to functions of two variables, t and u, or Ψ(t,u). We can introduce Ψ(t,u) into a double Fourier sine series [2],

$$\Psi(t,u) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} B_{lm} \sin\frac{l\pi t}{c_1} \sin\frac{m\pi t}{c_2},$$

Where,

$$B_{lm} = \frac{4}{c_1 c_2}\int_0^{c_1}\int_0^{c_2} \Psi(t,u) \sin\frac{l\pi t}{c_1} \sin\frac{m\pi t}{c_2} dt du.$$

Similarly, we can elaborate Ψ(t,u) into a double Fourier Cosine series

$$\Psi(t,u) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} B_{lm} \cos\frac{l\pi t}{c_1} \cos\frac{m\pi t}{c_2}.$$

The fundamentals of Fourier transforms are the next topic we will cover. The remaining portions of the paper are arranged as follows:

▪ Fourier Transform and their primaries results have been introduced.

▪ The use of the two-dimensional Fourier transform in image processing for several medical technologies has been described.

▪ We wrap up our study with a brief summary, recommendations for future research, and limitations.

1.1 Fourier Transform

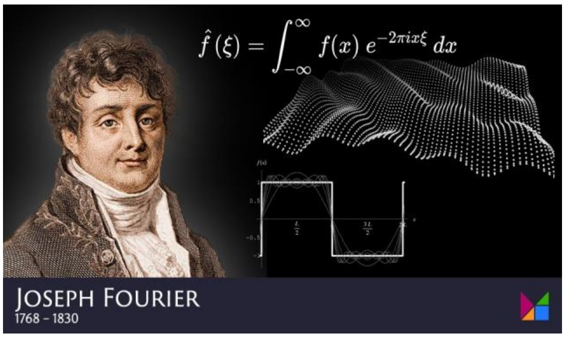

Figure 2: Definition of Fourier Transform.

A mathematical transformation also known as a Fourier Transform (FT) breaks down functions that rely on time or space into functions that simply depend on frequency. Using sine and cosine expressions with varying frequencies, a method called the Fourier Transform splits a waveform (a function or signal) into a distinct representation.

The Fourier Transform shows that any waveform may be expressed as the sum of sinusoidal components.

We discovered how to use the Fourier Series to transform any periodic function into a sum of sinusoids. The Fourier Transform is how this idea is applied to a non-periodic function. The frequency of oscillations is computed using the Fourier Transform. The Fourier Transform quantifies the amount of oscillations in the ϕ at the frequency ζ.

The Fourier Transform's mathematical representation is

$$\psi(\zeta) = \widehat{\Psi}(\zeta) = \int_{-\infty}^{\infty} \Psi(\eta)\, e^{-i\zeta\eta} d\eta.$$

While the Fourier series only applies to periodic signals, the Fourier Transform may be used for non-periodic signals. This is how the Fourier Transform and the Fourier Series differ from one another.

The followings are a few characteristics of the Fourier Transform :

1. The transformation is linear.
2. There is the property of time shift in the Fourier Transform.
3. There is the property of modulation in the Fourier Transform.
4. There is the duality in the Fourier Transform.

1.2 Fourier Integral

The Fourier integral is used to determine the Fourier transform. If a function $\Psi(\zeta)$ is piecewise smooth on all intervals $[-L,\ L]$ and satisfies the Dirichlet condition on all finite intervals, and if the integral $\int_{-\infty}^{\infty} |\Psi(\zeta)|\, d\zeta$ converges then

$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\Psi(\eta)|\, d\eta \int_{-\infty}^{\infty} \cos[v(\zeta-\eta)]\, dv$ is known as the Fourier Integral.

Cauchy's approach for solving partial differential equations is based on the Fourier Integral, which is highly helpful in the field of electrical communication.

1.2.1 Advantages of Fourier Transform

The breadth or magnitude of a Fourier transformation is represented in terms of points. The number of points used in the transformation represents both the frequency of values from the signal to be investigated and the number of values the conversion returned. The more points used in the transition, the higher the frequency resolution will be. The Fourier Transform preserves information in amplitude, harmonics, and phase while transferring the signal into the frequency domain using the whole waveform. The Fourier transformation preserves phase information, which allows the signal to be translated back into the time domain.The frequency components of a signal are a crucial feature that the Fourier Transform may disclose. By separating noisy or complicated data into a series of trigonometric or exponential functions, it seeks to improve the comprehensibility of the data. The Fourier transform is used to interpolate functions and facilitate communications. One of the most important jobs in signal processing is identifying frequency components, which is used extensively. It is employed to deconstruct a melodic composition's amplitude into the relative intensities of each of its constituent pitches. In engineering, it is employed to determine the primary frequencies in a vibration study.

1.3 Discrete Fourier Transform (DFT)

The discrete Fourier Transform is obtained by breaking down a set of integers into components with different frequencies.

To understand what the DFT does, consider an MP3 player that is linked to a speaker. The speaker receives audio data from the MP3 player in the form of voltage changes in electrical signals. These modifications cause the speaker drum to vibrate, which in turn moves air molecules and generates sound. An audio signal's changes over time can be shown as a graph, with time on the x-axis and the electrical signal's voltage or even the speaker's drum or air molecules' mobility on the y-axis.

The signal manifests as an irregular wave that resembles a squiggle in either scenario. But when we listen to the music that the squiggle produced, we can quickly recognize each instrument in a symphony orchestra since they were all playing different notes at the same time. When the component frequencies change in different directions, squiggle moves in the centre. It also moves upward when both of them travel downward. The DFT mathematically breaks down a signal into its individual frequencies, much like the human ear does physically. Among the significant uses of the Discrete Fourier Transform are

- Solving partial differential equations.
- Detection of targets from radar echoes.
- Correlation analysis.
- Computing polynomial multiplication.
- Spectral analysis.
- Convolutions and huge integer multiplications are examples of operations.
- Linear filtering etc.

1.4 Fast Fourier Transform (FFT)

It is feasible to significantly enhance the computation of the Discrete Fourier Transform and its Inverse Transform by matching even and odd functions together during computation. Combining even and odd functions, the Fast Fourier Transform [3] lowers the processing cost of the discrete Wavelet Transform from $O(N2)$ to $O(NlogN)$. Additionally, it features a Fast Fourier Transform in reverse. FFT is used to re-express an arbitrary function's Discrete Fourier Transform. The computing time of highly composite N is $O(NlogN)$ in the context of $N1$ smaller DFTs of sizes $N2$, successively, with composite size N = $N1N2$. After converting the DFT matrix into a product of sparse components—a matrix with a large number of zero entries—transformations are computed rapidly. The FFT is used to translate the data to the frequency domain and produce the results. Cooley and John Tukey are widely credited with creating the modern general fast Fourier transform technique in 1965.

1. Quick multiplication of polynomials and big numbers.
2. Efficient vector and matrix multiplication.
3. Filtering algorithms.
4. Quick algorithms for discrete cosine or sine transformations.
5. Fast Chebyshev approximation.
6. Calculation of isotope distributions.

2 FOURIER TRANSFORM IN BIOMEDICAL ENGINEERING [4]

Using the Fourier Transform, physical problems can be solved. The Fast Fourier Transform has several applications in the medical field. Several medical imaging modalities use the Fast Fourier Transform (FFT) for medical image de-noising, which creates images from raw data that has been acquired. The Fast Fourier Transform is a useful tool for streamlining frequency-domain signal analysis. The Fast Fourier Transform (FFT) is the fundamental tool used in digital signal processing. It is crucial for signal processing [5] tasks including compression, filtering, and de-noising.

2.1 2-D Image processing with FFT

Since we are primarily interested in digital pictures, we will limit our discussion to the Discrete Fourier Transform (DFT). Only a small number of frequency values are produced by the Discrete Fourier Transform (DFT), which operates by sampling the continuous Fourier Transform. However, in the spatial domain, this number is usually adequate to adequately characterise the image. When switching between the spatial and frequency domains, the picture size stays constant since the density of frequency components directly relates to the number of pixels in the original image.

For each pixel in the picture, a double summation is needed in order to calculate the values using the aforementioned formulae. However, this procedure may be made simpler because the Fourier Transform is separable. An intermediate representation is produced by first performing N one-dimensional Fourier Transforms along the spatial image's one dimension. The final frequency-domain picture is then created by applying a second set of N one-dimensional transformations along the opposite dimension. By transforming the two-dimensional transform into a series of two one-dimensional transforms, this method efficiently lowers the computational burden. The usual one-dimensional DFT computation is still technically demanding in spite of this optimization. This might be reduced to $Nlog2N$ if the one-dimensional DFTs are estimated using the Fast Fourier Transform (FFT). This is a significant improvement, particularly for large images. Most FFT versions restrict the size of the input image that may be altered to $N = 2n$, where n is an integer. The mathematical details are well described in the literature. The Fourier Transform produces a complex-valued picture as its output, which may be expressed using its magnitude and phase or by splitting it up into real and imaginary components. Since the magnitude provides the majority of the structural information about the spatial domain picture, it is usual practice in image processing to display simply the magnitude. To guarantee precise reconstruction, both the phase and magnitude information must be maintained if we want to do any frequency-domain operations before returning the picture to its original spatial form. The Fourier domain image has a far greater range than a picture in the spatial domain. As a result, in order to be appropriately exact, its values are frequently calculated and stored as float numbers.

2.2 Fourier Transform in medical imaging

An essential tool for image processing , the Fourier Transform divides an image into its sine and cosine components. The altered result depicts the original picture in the frequency domain, even if it exists in the spatial domain. frequency-based representation, where every point is associated with a certain frequency seen in the spatial picture. A mathematical definition of an image is some function $f(x, y)$, where y and x are the image's spatial coordinates and f (x, y) is the picture's brightness at the point (x, y).Computers employ digital images, where $f(x, y)$ is a function with numerical values for x, y, and brightness. Each component of a digital photo is called a picture element, or pixel for short, and the brightness is proportional to the grey level. A matrix of 256 by 256 or more grey levels might be present in a normal digital picture. The grey level is a representation of the amount of light that travels through a film that contains images. The Fourier Transform has several applications in medical imaging, including:

- Plain X-Rays.
- MRI (Magnetic Resonance Imaging).
- CT (Computed Tomography).
- CAT (Computerized Axial Tomography)
- Chest Radiography.

2.3 X-Ray

An X-ray [6,7] is a type of high-energy electromagnetic radiation that has penetrating qualities (X-radiation). The wavelengths of most X-rays range from 10 picometres to 10 nanometres, or 30 petahertz to 30 exahertz (310(16) Hz and 310(19) Hz) in frequency. X-rays are most frequently used to detect breast cancer, diagnose pneumonia, and check for fractures (broken bones). A 2-D projection of the 3-D object is provided by the X-ray as it is transmitted from the source. In essence, we then have two-dimensional pictures.

2.4 Chest Radiography

A radiographic projection of the chest used to detect issues affecting the chest, its contents, and surrounding tissues is called a chest radiograph, sometimes referred to as a chest X-ray (CXR) or chest film [8]. Chest radiographs are the most common film taken in medicine. A shift-invariant model of the two-dimensional point spread response functions of the scattered radiation is deconvolved using Fourier Transform techniques. Because it uses a digital radiograph that was taken using a standard chest imaging technique, no specialised imaging equipment is required.

Although the shift variant form of the scatter model is best suited for the lung field, suitable compensation is provided when the same model shape is applied to other chest regions. According to preliminary investigation, this technique can improve image contrast throughout the chest region. Using the FT and its inverse to remove extraneous information from an image:

1. An picture produced by fusing the similar Fourier spectrum with the sine wave and chest radiography images.
2. By altering the spatial frequency components, as shown by the darker regions in the image, the undesirable interference produced by the sinusoidal brightness pattern may be removed. The inverse FT is then used to reconstruct the original chest image, which is mainly undisturbed as seen in.

2.5 MRI

Radiologists frequently employ magnetic resonance imaging (MRI) as a diagnostic technique to see the inside organs and physiological functions of the body. It creates finely detailed pictures of organs and tissues using radiofrequency waves, gradient coils, and strong magnetic fields. MRI is frequently used in clinical settings and hospitals to diagnose illnesses, track the effectiveness of treatment, and identify the stage of certain ailments. When imaging soft tissues, MRI offers superior contrast compared to CT scans, which makes it particularly helpful for studying regions like the brain and abdomen.

A fundamental mathematical technique frequently used in signal processing, the Fourier transform is widely used in radiology and is crucial to the production of modern MR images. To understand MRI operations, one must have a basic understanding of the functions of the Fourier transform. A number of artefacts, such as k-space filling and MR image encoding, are based on the Fourier transform.

We may examine the frequency components of complicated signals using the Fourier Transform. The frequency domain allows for the visualisation and even manipulation of these signals. MR spectroscopy is a well-known clinical use of this, where frequency and amplitude spectra are used to present data. The existence and relative concentration of several biological metabolites within a given area of interest (ROI) are represented by each peak in this spectrum. Because their chemical structures differ, these metabolites resonate at slightly different frequencies.

2.6 MR spectroscopy

Instead of creating anatomical pictures, MR spectroscopy, in contrast to traditional MRI, is intended to analyse the chemical makeup of a targeted area of interest (ROI). It targets this more constrained region using a particular RF pulse bandwidth. Because of their various chemical structures, several neuronal metabolites, including myoinositol (ML), choline (Cho), creatine (Cr), glutamate and glutamine (Glx), N-acetyl aspartate (NAA), lactate (Lac), and lipids (Lip), resonate at different frequencies. A complicated mix of echoes from various metabolites makes up the signal that the ROI returns.
The Fourier Transform breaks down this signal into distinct frequency components and their corresponding amplitudes.

The word "relative" is crucial since the Fourier Transform only represents relative values, therefore the height of each peak in the MR spectroscopy spectrum only makes sense in relation to other peaks.

During the frequency and phase encoding stages, the magnetic field gradients (shown by open arrows in the top three photos) are gradually changed to sample spatial frequencies in this coronal brain slice. Even though there are just three examples provided, a wide variety of gradient setting combinations are needed to completely fill k-space. The inverse Fourier Transform, which basically combines the contributions of all spatial frequencies to generate the whole image, is used to rebuild the final image when k-space has been fully filled.

2.6. 1 Math of MRI

We must understand the mathematical basis of MRI picture creation. The spins of the hydrogen nuclei are aligned when we place a human or any other item to be scanned within a very big magnet. This now produces a net magnetisation, which is essentially what MRI aims to do. The magnetisation is first tipped into the transverse plane by a radio wave, and the position is then encoded by adding a magnetic field gradient. The magnetic field gradient changes the relative phase of these spins. $M(x)e^{i\phi}$.

Mathematically, we can represent that with an integral.

ϕ=$\int \omega dt$

=$\int \gamma B dt$

=$\int \gamma G.x dt$

=$x\int \gamma G dt$

=$x.k$

Finally, we get, $M(x)$=$e^{ix\cdot k}$.

The phase is the integral of the frequency over time as frequency is the change in phase over time. Sometimes a smooth linear gradient is all that B is. Assuming that the individual is stationary, x will not change over time.

2.7 Expandation of 2d illustration for brain image

First, we take the spring image's [9] whole transform. It will first be broken down into several parts, each of which is a stripped picture with a distinct spatial frequency. These parts will then be positioned on the k-space in accordance with their spatial frequencies. until every element has been accurately positioned on the k-space. After completing the Fourier Transform, we obtain the brain image's k-space.

On the other hand, we may get the brain picture from its k-space using the inverse Fourier transform. In order to determine their ease-based placements, the points on the k-space will first be transformed back into several scraped pictures with various spatial frequencies. The picture will then be restored by adding together all of these stripped images.

• The inner parts of k-space include the majority of an image's low-frequency components, whereas the outside portions contain the high-frequency components.

• The strapped patterns are always perpendicular to the line connecting a point at the centre of the k-space. Whereas the y-axis points correlate to horizontal patterns, the x-axis points correlate to vertical patterns.

Conclusion

Signal noise reduction, picture compression, MRI reconstruction utilizing FFT, and signal noise removal have all been thoroughly examined in this work. A noisy signal has been altered and a denoised signal has been retrieved using a single one-dimensional FFT. It may be used to produce melodic sounds and eliminate grating noises in noise cancellation software. Additionally, an auto encoder that denoises radio and ECG data is created using FFT. Here are other examples of how two-dimensional FFT affects visuals. With the aid of the FFT, the picture may be compressed up to a specific point without losing its identity. We observe that massive computations may be carried out with little storage by converting noisy data from the time domain to the frequency domain and lowering the amplitude of Fourier coefficients. The Fourier Transform has several uses in data science and other linear algebra packages as this concept has long been applied to data compression and sparse calculations.

This work inspires us to investigate the Fourier Transform's applicability in biomedicine and seismics.

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