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Spectral Analysis and Numerical Ranges in Banach Space and Measuring Chaos of Lorenz System

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Abstract: This paper represents the basic concepts spectrum analysis and numerical range dynamical system in Banach space. Li-Yorke chaos in Banach spaces is defined and their related theorems, definitions, lemmas, propositions, and corollaries are redefined, and some are proofed. Spectral and numerical analysis of chaotic dynamical systems are established through Lyapunov exponents, equilibrium etc. Stability analysis of dynamical system in Banach space is proofed that chaos has existed in chaotic systems. We have analysed the Lorenz system in Banach space with different equilibrium points and its stability are discussed. Measuring Chaos (Lorenz system) in Banach space are verified through basin of attraction, Poincare sections, strange attractors bifurcation analysis at different equilibrium points, and dissipativity natures and existence of the bounded attractor etc.

Keywords: Banach space, Chaos, Stability analysis, Lorenz system.

1. Introduction

Spectral theory and numerical range analysis play a fundamental role in functional analysis and operator theory. The spectrum of an operator provides deep insight into stability, invertibility, and long-term behavior, while the numerical range offers a powerful geometric tool for understanding operator properties. In Banach spaces, the numerical range enjoys convexity and strong connections with spectral theory via some theorems. However, extending these results to Banach spaces introduces additional challenges due to the lack of an inner product structure. This work aims to study matrix operators in both Banach space settings and to highlight the similarities and distinctions between their spectral and numerical properties ([1], [2], [3]).

Let X be a Banach space and B(X) be the algebra of bounded linear maps on X. Suppose $T \in$ B(X). The numerical range of T denoted by W(T) is:

$$W(T) := \{\langle Tf, f\rangle : f \in X, \|f\| = 1\}. \qquad (1)$$

Extensively studied the theory of the numerical range results, W(T) is a convex set [4]. Higher rank numerical range in the context of "quantum error correction" has been provided by Choi, Kribs and Życzkowski [5]. It is defined in such a way that Let $T \in$ B(X) and k ∈ $\mathbb{N}\infty := \mathbb{N} \cup \{\infty\}$. The k-rank numerical range of T, $\Lambda_k(T)$ is defined as

$$\Lambda_k(T) := \{\lambda \in C: PTP = \lambda\, P, \textit{for some projection } P \textit{ of rank k}\}, \qquad (2)$$

or, equivalently $\lambda \in \Lambda_k(T)$ if and only if there is an orthonormal set $\{f_j\}_{j=1}^{k}$ such that $\langle Tf_j, f_k\rangle = \lambda\delta_{j,k}$ for $j, k \in \{1,2,\dots,k\}$. Clearly,

$$W(T) = \Lambda_1(T) \supseteq \Lambda_2(T) \dots \supseteq \Lambda_k(T) \dots \quad . \quad (3)$$

Further, CKZ (Choi, Kribs and Życzkowski) have given a description of the higher rank numerical range of self-adjoint matrices in terms of its eigenvalues [6], [7]. Let Mn be the algebra of n-by-n matrices with complex entries. Suppose T ∈ Mn be a self-adjoint matrix and 1 ≤ k ≤ n. CKZ have provided

$$\Lambda_k(T) = [\, \lambda_{n-k+1,}\ {}_{\lambda_k}\, ,] \qquad (4)$$

where $\lambda_1 \gg \lambda_2 \gg \cdots \gg \lambda_n$are eigenvalues of T. They have putted forward a conjecture (CKZ conjecture) on the higher rank numerical range of normal matrices. Let $T \in M_n$ be a normal matrix and $1 \le k \le n$. CKZ conjecture says

$$\Lambda_k(T) = \cap_{1\le j_1<\cdots<j_{n-k+1}\le n} \mathrm{conv}\{\lambda_{j1,\dots,}\lambda_{jn-k+1}\},$$

(5)

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of T. Li, Poon and Sze have described the higher rank numerical range of a matrix as the intersection of the closed half plane determined by eigenvalue which settled CKZ conjecture in affirmative [10]. Let $T \in M_n$ and $1 \le k \le n$. They showed

$$\Lambda_k(T) = \cap_{\xi \in [0,2\pi)} \{\mu \in \mathbb{C}: \Re(e^{i\xi}\mu) \le \lambda_k\ (\Re(e^{i\xi}T))\} \qquad (6)$$

Woerdeman [8] have firstly shown the convexity of the higher rank numerical range of any operator independently. Let $T \in \mathrm{B(X)}$ be self-adjoint operator and E be the unique spectral measure associated with T. Suppose $\in N_\infty$. Let

$$A_k \coloneqq \{a \in \mathbb{R}: \dim \operatorname{ran} E(-\infty, a] < k\},$$
$$B_k \coloneqq \{b \in \mathbb{R}: \dim \operatorname{ran} E[b, \infty) < k\},$$
$$\Omega_k \coloneqq \{A_k^c \cap B_k^c\}.$$

Theorem 1.1. [9] Let $T \in \mathrm{B(X)}$ be self-adjoint operator and $k \in N_\infty$. Then

$$\Lambda_k(T) = \cap_{V \in V_k}\ W(V^*TV) = \Omega_k, \qquad (7)$$

where Λ_k be the set of all isometries $V: X \to X$ such that codimension of range V is less than k.

In other words, the above theorem states that $\lambda \in \Lambda_k(T)$ if and only if dim ran $E(-\infty, \lambda] \ge k$ and dim ran $E[\lambda, \infty] \ge k$ for $k \in N_\infty$ and self-adjoint T.

Later, Li, Poon and Sze have extended these concepts ([10], [11]) for any operator. Suppose $T \in \mathrm{B(X)}$ and $k \in \mathbb{N}$. Let

$\Lambda_k(T) = \cap_{\xi \in [0,2\pi)} \{\mu \in \mathbb{C}: \Re(e^{i\xi}\mu) \le \lambda_k\ (\Re(e^{i\xi}T))\}$ Where $\lambda_k(X) \coloneqq \sup\{\lambda_k(V^*TV): V: \mathbb{C}^k \to \mathbb{X}$ such that $V^*V = I\}$ for some self-adjoint operator $T \in \mathrm{B(X)}$. Let **Int**(S) denote the relative interior of S and $\bar{S}$ denote the closure of S where $\subseteq \mathbb{C}$. Li et al. showed the following:

Theorem 1.2. [12] Let $T \in \mathrm{B(X)}$ and $k \in \mathbb{N}$. Then

$$\mathrm{Int}(V_k(T)) \subseteq \Lambda_k(T) \subseteq V_k(T) = \overline{V_k(T)}.$$

An operator T acting on a Banach space X is called Li-Yorke chaotic if there exists an uncountable set $\Gamma \subset X$ with the following property: For any two different points $x, y \in \Gamma$, the sequences $\{T^n x\}$ and $\{T^n y\}$ behave in a mixed way. More precisely,

$$\lim_{n\to\infty} \inf\|T^n x - T^n y\| = 0 \text{ and } \lim_{n\to\infty} \sup\|T^n x - T^n y\| > 0. \qquad (8)$$

In simple way, this means that the distance between the two trajectories gets arbitrarily close to zero infinitely often but also becomes separated by some positive amount infinitely. Each pair (x, y) with this behavior is called a Li–Yorke pair ([12], [13]). This concept of chaos was first introduced by Li and Yorke in 1975 while studying dynamical systems on intervals. It was the first formal mathematical definition of chaos and later became widely used in dynamical systems and operator theory [14].

Motivated by the above discussion, we have studied the spectral analysis and numerical range of chaotic systems in Banach space. We have established fundamental concepts of chaotic dynamical systems within different mathematical branches such as Banach spaces and topological spaces. Dynamical systems in Banach spaces have been studied regarding the existence of strange attractors, equilibria, and linearization, stability theorems, numerical range, and transient growth. Measurement of chaos in Banach-Space Lorenz Systems has been analyzed with tools such as Lyapunov exponents, strange attractors, basins of attraction, Poincaré sections, and dissipative.

The structure of the paper is as follows: Section 1 introduces the topic. Section 2 outlines fundamental concepts of the existence of dynamics in Banach Space. Section 3 establishes spectral analysis and numerical ranges. Section 4 presents chaotic dynamical systems and stability theory. Section 5 discusses chaos measurement in Lorenz systems, and Section 6 concludes the study.

2 Fundamental Concepts of Existence of Dynamics in Banach Space

Let $\mathrm{B(X)}$ be a separable Banach space over the field $\mathbb{K}(\mathbb{R}\ or\ \mathbb{C})$ with norm $\|\cdot\|$ and $T : X \to X$ be a bounded linear operator. A transitive point in X is usually called a hypercyclic vector for T, and we say that T is hypercyclic if it has some hypercyclic vector [21]. Let (X) be the collection of bounded operators. The strong operator topology (SOT) on $\mathrm{B(X)}$ is defined as follows: any $T \in \mathbb{B}(X)$ has a neighborhood basis consisting of sets of the form

$$V_{x_1,\ldots,x_n,\varepsilon,T} = \{S \in \mathrm{B(X)} : \| Sx_\mathrm{i} - Tx_\mathrm{i} \| < \varepsilon, i = 1,\ldots,n\}, \quad (9)$$

where $x_1,\ldots,x_n \in X$ and $\varepsilon > 0$. DULYC(X) means the collection of all densely uniformly Li-Yorke chaotic operators in $\mathrm{B(X)}$ [13]. Every weakly mixing operator is densely uniformly Li-Yorke chaotic.

Proposition 2.1. Let X be an infinite-dimensional separable Banach space. Then DULYC(X) is dense in $\mathrm{B(X)}$ with respect to the strong operator topology (SOT).

If X is a separable Hilbert space, the strong* operator topology (SOT*) on $\mathrm{B(X)}$is defined by neighborhood bases of the form:

$$V^*_{x_1,\ldots,x_n,\varepsilon,T} = \{S \in B(X) : \| Sx_\mathrm{i} - Tx_\mathrm{i} \| < \varepsilon, \| S^* x_\mathrm{i} - T^* x_\mathrm{i} \| < \varepsilon, i = 1,\ldots,n\}. \quad (10)$$

Proposition 2.2. Assuming that X is an infinite-dimensional separable Hilbert space and $M > 1$. Then DULYC(X) $\cap B_M(X)$ is a SOT $*$-dense $G_\delta(X)$ subset of $B_M(X)$.

Proof:

Claim: DULYC(X) $\cap$ $B_M(X)$.is a $G_\delta$subset of $\mathbb{B}_M(X)$ with respect to SOT.

Proof of Claim: Let $\mathcal{U}$ be a countable topological basis of X. Without loss of generality, assume that open sets in $\mathbb{U}$ are nonempty. Fixed $k \geq 2, N \in N$ and $_1,\ldots,U_n \in \mathcal{U}$. Let $P(U_1,\ldots,U_k,N) = \{T \in \mathrm{B}_M(\mathrm{X}) : \exists x_i \in U_i$ for $i = 1,\ldots,k$ and $n \geq N$ s.t. $\|T^n(x_i - x_j)\| < \frac{1}{N}$ for $1 \leq i < j \leq k\}$ and $D(U^1,\ldots,U_k,N) = \{T \in B_M(\mathrm{X}) : \exists x_i \in U_i$ for $i = 1,\ldots,k$ and $n \geq N$ s.t. $\|T^n(x_i - x_j)\| < N$ for $1 \leq i < j \leq k\}$.

For every $n \in N$, the map (B_M(X):, SOT) $\to$ ($B(X)$, SOT), $T \mapsto T^n$ is continuous [15]. It is easy to check that $P(U_1,\ldots,U_k,N)$ and $D(U_1,\ldots,U_k,N)$are open in (B_M(X), SOT). Now by the above theorem,

DULYC(X) $\cap B_M(\mathrm{X}) = \bigcap_{k=2}^{\infty} \bigcap_{U_1,\ldots,U_k \in U} \bigcap_{N=1}^{\infty} P(U_1,\ldots,U_k,N) \cap D(U_1,\ldots,U_k,N)$.
This shows that DULYC(X) $\cap\ \mathbb{B}_M(\mathrm{X})$ is a G_δ subset of $\mathrm{B(X)}$.

Since the strong$*$ operator topology is finer than the strong operator topology, by the Claim DULYC(X) $\cap B_M(\mathrm{X})$ is a G_δ subset of $B_M(\mathrm{X})$ with respect to the strong$*$ operator topology. The collection of weak mixing operators in $\mathrm{B_M(X)}$ is SOT^* -dense in $B_M(\mathrm{X})$. Hence, DULYC(X) $\cap$ $B_M(\mathrm{X})$ is a SOT^* -dense G_δ subset of $B_M(\mathrm{X})$ ([23], [24]).

Lemma 2.3. Let $T \in \mathrm{B(X)})$ and for every $k \geq 2$, Pro$x_k(f)$ is dense in X^k if and only if there exist a sequence $\{p_n\}$ in N and a dense subset K of X such that for every $x \in K$, $\lim_{n\to\infty} T^{p_n}x = 0$.

Proof: Suppose that Pro$x_k(f)$ is dense in X^k for every $k \geq 2$. Since, we know that there exists a sequence $\{p_n\}$ in N and a dense subset S of X such that for any pair $x, y \in S$ we have

$$\lim_{n\to\infty} \|T^{p_n}x - T^{p_n}y\| = 0.$$

Fix $y \in S$ and $let\ K = S - y$. Then K is a dense subset of X and for any $x \in K$,

$$\lim_{n\to\infty} T^{p_n}x = 0.$$

Now suppose that there exists a sequence $\{p_n\}$ in N and a dense subset K of X such that for every $x \in K$, $\lim_{n\to\infty} T^{p_n}x = 0$. Then for any $x_1, x_2 \in K$ with $x_1 \neq x_2$ we have

$$\lim_{n\to\infty} \|T^{p_n}x_1 - T^{p_n}x_2\| \leq \lim_{n\to\infty} \|T^{p_n}x_1\| + \lim_{n\to\infty} \|T^{p_n}x_2\| = 0,$$

which completes the proof of the lemma.

Let (X, d) be a metric space and $f: X \rightarrow X$ be a continuous map. We can say that f has sensitive dependence on initial conditions or is sensitive briefly if there exists some $\delta > 0$ such that for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ and some $n \in N$ such that $d(f^{n}(x), f^{n}(y)) > \delta$.

We have the following characterization of sensitive operators on a Banach space [19], [22], [23].

Lemma 2.4. Let $T \in B(X)$. Then the following assertions are equivalent:

1. T is sensitive;
2. $\sup_{n} \| T^{n} \| = \infty$;
3. there exists a vector $x \in X$ such that $\lim \sup_{n\to\infty} \| T^{n}x \| = \infty$;
4. the set of all vectors x ∈ X satisfying $\lim \sup_{n\to\infty} \| T^{n}x \| = \infty$ is residual in X.

Let $T \in B(X)$. A vector subspace Y of X is called an irregular manifold for T if every non-zero vector $y \in Y$ is irregular for T. According to the above theorems, we have the following sufficient criterion for the existence of a dense irregular manifold.

Theorem 2.5. Let $T \in B(X)$. If T is sensitive and there exists a sequence $\{p_n\}$ in N and a dense subset K of X such that for every $x \in K, \lim_{n\to\infty} \| T^{n}x \| = 0$, then T admits a dense irregular manifold. Combing Lemmas 2.3 and 2.4 and applying above theorems, we have the following consequence.

Corollary 2.6. Let $T \in B(X)$. If T is densely uniformly Li-Yorke chaotic, then T admits a dense irregular manifold. We have the following sufficient conditions of dense uniform Li-Yorke chaos for operators.

Theorem 2.7. If $T \in B(X)$. If T is sensitive and $\{x \in X : , \lim_{n\to\infty} T^{n}x = 0\}$ is dense in X, then it is **densely uniformly Li-Yorke chaotic.**

Proof: By Lemma 2.3, we have for every $k \geq 2$, $\mathrm{Prox}_k(f)$ is dense in X^{k}. it is sufficient to show that for every $k \geq 2$, $D_k(f)$ is dense in X^{k}. Fix $k \geq 2$ and nonempty open subsets $U_1, \ldots, U_k$ of X. Pick points $x_i \in U_i$ such that, $\lim_{n\to\infty} T^{n}x_i = 0$ for $i = 1, \ldots, k$. By Lemma 2.4, pick a point $z \in X$ such that

$\mathrm{Lim}\ sup_n \| T^{n}z \| = \infty$;. Pick distinct scalars $\alpha_1, \ldots, a_k \in K$ such that $x_i + \alpha_i z \in U_i$ for $i = 1, \ldots, k$. Then $(x_1 + \alpha_1 z, \ldots, x_k + a_k z) \in D_k(f) \cap U_1 \times \ldots \times U_k$. This shows that $D_k(f)$ is dense in X^{k}.

Theorem 2.8. Let $T \in B(X)$, kernel of T is $\ker(T) = \{x \in X: T\,x = 0\}$ and generalized kernel of T is $\bigcup_{n\in N} \ker(T^{n})$. According to Theorem 2.7, if the generalized kernel of T is dense in X, then T is sensitive if and only if it is densely uniformly Li-Yorke chaotic [13].

To apply Theorem 2.7, we will consider the unilateral backward weighted shift $B_w : X \rightarrow X$, $(x_1,\ x_2, \ldots) \mapsto (w_2x_2,\ w_3x_3, \ldots)$ on $X = l^{p}(N),\ 1 \leq p < \infty$ or $X = c_0(N)$, where $w = \{w_j\}_{j\in N}$ is a bounded sequence of non-zero weights. It is easy to verify that for each $n \in N$, $\|B_w^n\| = \sup_{k\in N} \prod_{j=k+1}^{n+k+1} |w_j|$. For a unilateral backward weighted shift Bw it is Li-Yorke chaotic if and only if $\sup_{k,n\in N} \prod_{j=k+1}^{n+k+1} |w_j| = \infty$. The generalized kernel of B_w is the collection of sequences with only finite non-zero coordinates, which is dense in X. By Theorem 2.7, we have the following result.

Corollary 2.9. Let B_w be a unilateral backward weighted shift on $X = l^{p}(N),\ 1 \leq p < \infty$ or $X = c_0(N)$ with the weight sequence $w = \{w_j\}_{j\in N}$. Then the following conditions are equivalent:

1. $\sup_{k,n\in N} \prod_{j=k+1}^{n+k+1} |w_j| = \infty$.
2. there exists a Li-Yorke scrambled pair;
3. B_w is Li-Yorke chaotic;
4. B_w admits a dense irregular manifold;
5. B_w is densely uniformly Li-Yorke chaotic.

Now we consider the bilateral backward weighted shift $B_w : X \rightarrow X$, $(\ldots, x_{-1}, [x_0], x_1, \ldots) \mapsto (w_0x_0,\ [w_1x_1],\ w_2x_2,\ , \ldots)$ on $X = l^{p}(N),\ 1 \leq p < \infty$ or $X = c_0(N)$ with the weight sequence $w = \{w_j\}_{j\in N}$ is a bounded sequence of non-zero weights. It is easy to verify that for each $\|B_w^n\| = sup_{k\in N} \prod_{j=k+1}^{n+k+1} |w_j|$.

Note that in this case the generalized kernel of B_w is trivial, so we cannot apply Theorem 2.7 directly. We have the following characterization of dense uniform Li-Yorke chaos for bilateral backward weighted shifts.

Theorem 2.10. Let B_w be a bilateral backward weighted shift on $X = l^p(N)$, $1 \leq p < \infty$ or $X = c_0(N)$ with the weight sequence $w = \{w_j\}_{j\in N}$. Then the following conditions are equivalent:

1. $\lim\ \inf_{n\to\infty} \prod_{j=-n+1}^{0} |w_j| = 0$ and $\sup_{k,n\in N} \prod_{j=k}^{k+n} |w_j| = \infty$;

2. there exists a Li-Yorke scrambled pair;

3. B_w is Li-Yorke chaotic;

4. B_w admits a dense irregular manifold;

5. B_w is densely uniformly Li-Yorke chaotic.

Proof: For simplicity, we only prove the case $X = l^1(N)$, the proofs of other cases are similar. $(5) \Rightarrow (3) \Rightarrow (2)$ are obvious and $(5) \Rightarrow (4)$ follows from Corollary 2.6. $(2) \Rightarrow (1)$. Let (x, y) be a Li-Yorke scrambled pair and let $z = x - y$. Then $z \neq 0$. Pick $k \in N$ such that $zk \neq 0$. For every $n \in N$, $\| B_w^n z \| \geq |zk|.\prod_{j=k-n+1}^{n} |w_j|$.

As $\lim\ \inf_{n\to\infty} \| B_w^n z \| = 0$, one has $\lim\ inf_{n\to\infty} | \prod_{j=k-n+1}^{k} w_j = 0|$. Since the weights are not equal to zero, this is equivalent to

$\lim\ \inf_{n\to\infty} \prod_{j=-n+1}^{0} |w_j| = 0$. To show that

$\sup_{k,n\in N} \prod_{j=k}^{k+n} |w_j| = \infty$; by Lemma 2.4 it is sufficient to show that B_w is sensitive. Let $\delta = \lim \sup_{n\to\infty} \| B_w^n z \|$. Then $\delta > 0$. If $\delta = \infty$, then by Lemma 2.4 B_w is sensitive.

Now assume that $\delta < \infty$. Fix $x \in X$ and $\varepsilon > 0$. There exist $n_1, n_2 \in N$ with $n_1 > n_2$ such that

$\|B_w^{n_1}\| < \varepsilon$ and $\|B_w^{n_2}\| > \delta/2$. Then

$\| x - (x + B_w^{n_1} z) \| < \varepsilon$ and

$$\| B_w^{n_2-n_1} x - B_w^{n_2-n_1}(x + B_w^{n_1} z) \| = \|B_w^{n_2}\|. > \delta/2$$

This implies that B_w is sensitive. $(1) \Rightarrow (5)$. Let $\{e_i\}_{i\in N}$ be the canonical basis of $l^1(N)$. Note that by the definition, one has $B_w e_i = w_i e_{i-1}$ for all $i \in N$. There exists a sequence $\{p_n\}$ in N such that $\lim_{n\to\infty} \prod_{j=-p_n+1}^{0} |w_j| = 0$. Then for every $s \geq 1$, $\lim_{n\to\infty} \|B_w^{p_n+s} e_k\| = \lim_{n\to\infty} \prod_{j=-p_n+1}^{s} |w_j| = 0$.

If $x = \sum_{j=-s}^{s} \alpha_j e_j$, then

$lim_{n\to\infty} \|B_w^{p_n+s} x\| \leq lim_{n\to\infty} \sum_{j=-s}^{s} |\alpha_j|$ $. \|B_w^{p_n+s} e_j\| =$ $\lim_{n\to\infty} \sum_{j=-s}^{s} |\alpha_j|\ . \|B_{w_j}^{p_n+s}(B_w^{s-j} e_s)\| \leq$ $\lim_{n\to\infty} \sum_{j=-s}^{s} |\alpha_j|. \|B_w^{s-j}\|. \|B_w^{p_n+s} e_s\| = 0.$

Fix $k \geq 2$ and nonempty open subsets $U_1, \ldots, U_k$ of X. There exists $s \in N$ and $x_i \in U_i$ such that

x_i can be expressed as $\sum_{j=-s}^{s} \alpha_j^{(i)} e_j$ for $i = 1, \ldots, k$. As $\lim_{n\to\infty} \|B_w^{p_n+s} x_i\| = 0$ for $i = 1, \ldots, k, (x_1, \ldots, x_k)$ $\text{pro}x_k(B_w) \cap U_1 \times \cdots \times U_k$ This shows that $\text{Pro}x_k(B_w)$ is dense in X_k.

By $\sup_{m,n\in N} \prod_{j=m}^{m+n} |w_j| = \infty$ and the weight sequence $\{w_j\}$ is bounded, there exist two sequences $\{q_n\}$ in N and $\{m_n\}$ in N such that $q_n \to \infty$, $|m_n| \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} \prod_{j=m_n}^{m_n+q_n} |w_j| = \infty$.

We have two cases (1) $m_n > 0$ for all n and (2) $m_n < 0$ for all n. We only consider the case (1), as the case (2) is similar. Now assume that

$m_n > 0$ for all n. Then $\lim_{n\to\infty} \|B_w^{q_n+s} x_{m_n+q_n}\| = lim_{n\to\infty} \prod_{j=m_n}^{m_n+q_n} |w_j| = \infty$. Fix $k \geq 2$ and nonempty open subsets $V_1, \ldots, V_k$ of X. Then there exist $s \in N$ and $y_i \in U_i$ such that y_i can be expressed as $\sum_{j=-s}^{s} \beta_j^{(i)} e_j$,

for $i = 1, \ldots, k$. There exists $\varepsilon > 0$ such that $y_i + \alpha e_j \in U_i$ for $i = 1, \ldots, k$, $j > s$ and $|\alpha| < \varepsilon$.

For each $n \in N\ and\ i = 1, \ldots, k$,

$$z_i = y_i + \frac{ii}{k+1} e_{m_n+q_n}.$$

Then

$\lim_{n\to\infty} \|B_w^{q_n}(z_i - z_j)\| \geq \lim_{n\to\infty} \frac{1}{k+1} \|B_w^{q_n} e_{m_n+q_n}\| = \infty.$

This show $(z_1, \ldots, z_k) \in D_k(B_w) \cap V_1 \times \cdots \times V_k$. Hence $D_k(B_w)$ is dense in X^k. By the above theorem B_w is densely uniformly Li-Yorke chaotic. According to Theorem 2.10 and Lemma 2.3, we have the following consequence about the

property of sequences in K, which should be of independent interest.

Corollary 2.11. Let $w = \{w_j\}_{j\in N}$. be a bounded non-zeros sequence in K. If $\lim\ inf_{n\to\infty} \prod_{j=-n+1}^{k} |w_j| = 0$, then there exists a sequences $\{p_n\}$ in N such that $\lim_{n\to\infty} \prod_{j=-p_n+k+1}^{0} |w_j| = 0$ for all $k \in N$.

Proof: We consider the bilateral backward weighted shift $B_w : l^1(N) \to l^1(N)$, $B_w e_i = w_i e_{i-1}$ for all $i \in N$. By the proof of $(1) \Rightarrow (5)$ of Theorem 2.10, For every $k \gg 2$, $\text{Pro}x_k(B_w)$ is dense in $l^1(N)^k$.

By Lemma 2.3, there exist a sequence $\{p_n\}$ in N and a dense subset K of $l^1(N)$, such that for every

$x \in K$, $\lim_{n\to\infty} B_n^{p^n} x = 0$. Fix $k \in N$. As K is dense $l^1(N)$, there exists $x \in K$ such that $\|x - e_k\| < 1$. In particular, $x_k \neq 0$. Then $\lim_{n\to\infty} \prod_{j=-p_n+k+1}^{k} |w_j| = \lim_{n\to\infty} \|B_w^{p_n} e_k\| \leq \frac{1}{|x_k|} \lim_{n\to\infty} \|B_w^{p_n} x\| = 0.$

This ends the proof

We construct an operator T on $l^2(N)$, such that the whole space $l^2(N)$, is a uniformly Li-Yorke scrambled set. By linearity, we have the following result: the existence of a "large" uniformly Li-Yorke scrambled set implies the whole space is uniformly Li-Yorke scrambled. Recall that a subset A of X is said to be locally residual if there exists a nonempty open subset U of X such that $A \cap U$ is residual in U.

Proposition 2.12. Let $T \in \mathcal{B}(X)$.. If there exists some uniformly Li-Yorke scrambled subset of Xwhich is locally residual, then the whole space X is uniformly Li-Yorke scrambled.

Proof: Let S be a uniformly Li-Yorke scrambled subset of X which is locally residual. There exist two sequences $\{p_n\}$ and $\{q_n\}$ in N such that for any pair $x, y \in S$ with $x \neq y$, we have

$\lim_{n\to\infty} \|T^{p_n}x - T^{p_n}y\| = 0$ and $\lim_{n\to\infty} \|T^{q_n}x - T^{q_n}y\| = \infty$.

As S is locally residual, there exists a vector $z \in X$ and $\delta > 0$ such that $S \cap B(z, 2\delta)$ is residual in $B(z, 2\delta)$. Fix a non-zero vector $x \in X$ and pick $\varepsilon > 0$ such that $\| \varepsilon x \| < \delta$. By linearity,

$(S + \varepsilon x) \cap B(z + \varepsilon x, 2\delta)$ is residual in $B(z + \varepsilon x, 2\delta)$. Note that $B(z, \delta) \subset B(z + \varepsilon x, 2\delta)$, then $(S + \varepsilon x) \cap S \cap B(z, \delta)$ is residual in $B(z, \delta)$. Pick a point $y \in (S + \varepsilon x) \cap S$. Then $y, y - \varepsilon x \in S$ and therefore $\lim_{n\to\infty} \|T^{p_n}x\| = 0$ and $\lim_{n\to\infty} \|T^{q_n}x\| = \infty$. This implies that the whole space X is uniformly Li-Yorke scrambled.

3 Spectral and Numerical Analysis of Chaotic Dynamical Systems

Consider a nonlinear discrete-time dynamical system $x_{n+1} = F(x_n),\ x_n \in X,$ (11) where X is a Banach or Hilbert space and $F : X \to X$ is a nonlinear mapping. Chaotic behavior is characterized by sensitivity to initial conditions, topological mixing, and dense periodic orbits. To analyze such systems using spectral theory, Koopman operator K defined on a suitable function space X and $g: X \to C$ by

$$(\kappa g)(x) = g(F(x)). \qquad (12)$$

Although the underlying system is nonlinear, the Koopman operator is linear (but infinite-dimensional).

3.1 Spectrum of Operators Associated with Chaotic Dynamics

Let T be a bounded linear operator on a Banach space X. The spectrum of T is defined as $\sigma(T) = \{\lambda \in C: T - \lambda I \text{ is not invertible}\}$. (13)

For chaotic systems, the associated operator (Jacobian, Koopman, or Perron–Frobenius operator) often establishes the continuous or residual spectrum, eigenvalues inside and on the unit circle, spectral radius equal to one.

3.1.1 Lyapunov Exponents and Spectrum

Let $DF(x)$ denote the Frechet derivative of F. The linearized system along a trajectory is

$$v_{n+1} = DF(x_n)v_n. \qquad (14)$$

Lyapunov exponents $\{\lambda_i\}$ characterize exponential growth rates and are related to the logarithm

of the spectral radius:

$$\max_i \lambda_i \lim_{n\to\infty} \frac{1}{n} \log r\left(\prod_{k=0}^{n-1} DF(x_k)\right), \qquad (15)$$

where $r(\cdot)$ denotes the spectral radius. A positive Lyapunov exponent implies spectral instability and chaotic behavior.

3.2 Numerical Range of Operators in Chaotic Systems

For a bounded linear operator T on a Hilbert space H, the numerical range is defined by

$$W(T) = \{\langle T\,x, x\rangle : x \in H, \| x \| = 1\}. \qquad (16)$$

The numerical range of L is

$$W(L) = \{\langle Lx, J(x)\rangle : \| x \| = 1\}. \qquad (17)$$

In general, the numerical range in Banach spaces, is defined via semi-inner products or duality map-pings. Let $J(x) \in X^*$ denote the normalized duality map. The numerical range provides geometric insight into operator behavior and satisfies:

$$\sigma(T) \subset \overline{W(T)}. \qquad (18)$$

In chaotic systems, numerical ranges often represent the non-convex for non-normal operators, extend beyond the spectral radius, and indicate transient growth even when asymptotically stable.

4 Lorenz-Type Systems in Banach Spaces and Stability Theory

Let X be a real or complex Banach space with norm $\| \cdot \|$. We consider an abstract Lorenz-type dynamical system of the form

$$\frac{d}{dt}X(t) = Ax(t) + F(x(t)), \qquad (\mathbf{19})$$

where, $A: D(A) \subset X \to X$ is a (possibly unbounded) linear operator, $F: X \to X$ is a nonlinear operator, typically quadratic or bilinear, $X(t) \in X$ represents the system state. Equation (19) generalizes the classical Lorenz system by allowing infinite-dimensional state spaces, as arise in fluid dynamics, and distributed parameter systems. They are essential in stability analysis of equilibria and modeling chaos in fluid dynamics, population models, and control systems. Thus, Banach spaces form the foundational setting for spectral analysis, numerical range theory, and nonlinear dynamics.

4.1 Equilibria and Linearization in Banach Spaces

Let x^*∈ X be an equilibrium point of (1**9**), that is,
Ax^*+ $F(x^*)$ = 0. (20)

To analyze the local behavior near x^*, we consider the linearization of the system about this equilibrium. The linearized equation is written as: $\frac{d}{dt}x(t) = Lx(t)$, and $L = A + DF(x^*)$, $(\mathbf{21})$

and $DF(x^*)$ denotes the Frechet derivative of F at the point $\boldsymbol{x}^*$. The linear operator $L : D(L) \subset X \to X$ determines the local stability properties of the equilibrium $\boldsymbol{x}^*$. In particular, the spectral properties of $\boldsymbol{L}$ characterize whether perturbations near $\boldsymbol{x}^*$ decay, persist, or grow with time.

Theorem 4.1 (Spectral Stability in Banach Spaces). Assume that L generates a co-semigroup $\{T(t)\}_{t\geq 0}\, on\, X$, the spectrum of L satisfies $\sup_{\lambda\in\sigma(L)} Re(\lambda) < 0$.Then the equilibrium x^* is locally exponentially stable.

Proof: By applying the Hille–Yosida theorem, L generates an exponentially stable semigroup: $\| T(t) \| \leq Me^{-\omega t}, \omega > 0$. Standard nonlinear perturbation results imply that solutions of the nonlinear system remain close to x^* and decay exponentially for sufficiently small initial perturbations.

Theorem 4.2 (Spectral Instability). If there exists $\lambda_0 \in \sigma(L)$ such that $Re(\lambda_0) > 0$, then the equilibrium x^* is unstable. This condition generalizes the classical Lorenz instability criterion to Banach spaces.

Theorem 4.3 (Numerical Range Growth Criterion). If sup $Re(W(L)) > 0$, then the linearized system exhibits transient growth of perturbations, even if $\sup \lambda \in \sigma(L) Re(\lambda) < 0$. This phenomenon explains short-time instability and sensitivity to initial conditions in Lorenz-type systems.

5 Measurement of Chaos in Banach Space Lorenz Systems

Lorenz-type systems in Banach spaces exhibit chaos when the linear operator L is non-normal and its numerical range extends into the right half-plane. This leads to transient growth, amplified by nonlinear effects, and ultimately results in chaotic dynamics. Such mechanisms are central to turbulence, three-dimensional chaotic attractors, and applications in climate and fluid convection.

Consider the three–dimensional Lorenz system:

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = x(\rho - z) - y,$$
$$\dot{z} = xy - \beta z, \qquad (22)$$

where the parameters σ, ρ, $\beta > 0$. Taking, $\sigma = 10, \rho = 28, \beta = {}^{8}\!/_{3}$.

5.1 Determination of Equilibrium Points

Equilibrium points satisfy $\dot{x} = \dot{y} = \dot{z} = 0$.

Thus, $\sigma(y - x) = 0,$

$$x(\rho - z) - y = 0,$$

$$xy - \beta z = 0. \qquad (23)$$

The Lorenz system (22) has one trivial equilibrium, $E_0 = (0, 0, 0)$, and two symmetric equilibria (existing when $\rho > 1$),

$E_{\pm} = \left(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho - 1\right)$.
$E_{\pm} = (\pm\, 8.485,\ \pm 8.485,\ 27)$

5.2 Lyapunov Exponents and Poincare Section of the Lorenz System

Let $X(t) = (x(t), y(t), z(t))^T$ be a trajectory of the Lorenz system (22). The evolution of an infinitesimal perturbation $\delta X(t)$ is governed by the variational equation:

$$\delta \dot{X} = J(X(t))\,\delta X, \quad (24)$$

where $J(X)$ is the Jacobian matrix given by

$$J(X) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}. \quad (25)$$

The Lyapunov exponents $\lambda_i = \lim_{n\to\infty} \frac{1}{t} \ln \frac{\|\delta Xi\,(t)\|}{\|\delta \boldsymbol{Xi}\,(0)\|}$. (26)

For the classical parameter values $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$, the Lyapunov spectrum is approximately, $\lambda_1 \approx 0.905, \lambda_2 \approx 0, \lambda_3 \approx -14.57$. Since $\lambda_1 > 0$, the Lorenz system (22) exhibits sensitive dependence on initial conditions, which lead to be the benchmark of chaos. The sum of Lyapunov exponents satisfies

$$\lambda_1 + \lambda_2 + \lambda_3 = -(\sigma + 1 + \beta), \quad (27)$$

which reflects the dissipative nature of the Lorenz system (22).

Let the Lyapunov exponents of the chaotic system be ordered as: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We define an integer j such that $\sum_{j=1}^{j} \lambda_j \geq 0$ and $\sum_{j=1}^{j+1} \lambda_j < 0$.

The Kaplan-Yorke (Lyapunov) dimension is:

$$D_{KY} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|} \approx 2 + \frac{\lambda_1}{|\lambda_3|} = 2.06 \quad (28)$$

5.3 Poincare Section

To further analyze the geometric structure of the attractor, a Poincare section is introduced. Let the section be defined by the plane,

$$\Pi = \{(x, y, z) \in R^3 : z = \rho - 1\}. \quad (29)$$

The Poincare map $P : \Pi \to \Pi$ is defined as

$$P(X_n) = X_{n+1}, \quad (30)$$

where X_n is the nth intersection of the trajectory with the plane Π in a fixed orientation.

The discrete sequence $\{X_n\}$ reveals the underlying stretching and folding mechanism of the Lorenz attractor (22). The resulting return map often exhibits a unimodal structure, similar to the one-dimensional chaotic maps. The existence of a strange attractor is supported by a positive Lyapunov exponent, Fractal (non-integer) Kaplan–Yorke dimension, and complex structure of the Poincare return map.

5.4 Basin of Attraction of the Lorenz System

Let $\phi_t(X_0)$ denote the flow generated by the Lorenz system (22) with initial condition $X_0 \in R^3$. The basin of attraction of an attractor $\mathcal{A}$ is defined as

$$B(\mathcal{A}) = \{X_0 \in R^3 : \lim_{n\to\infty} dist(\phi_t(X_0,\ \mathcal{A})\}. \quad (31)$$

In other words, $B(\mathcal{A})$ consists of all initial conditions whose trajectories asymptotically approach the attractor $\mathcal{A}$.

5.5 Dissipativity and Existence of a Bounded Attractor

The divergence of the Lorenz vector field is

$$\nabla \cdot F = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\sigma - 1 - \beta < 0. \quad (32)$$

Since the divergence is negative, the Lorenz system is dissipative. This guarantees the existence of a bounded attractor for sufficiently large ρ. Consequently, volumes in phase space contract exponentially:

$$V(t) = V(0)\, e^{-(\sigma+1+\beta)t}. \quad (33)$$

This implies the existence of a bounded absorbing set $D \in R^3$ such that all trajectories eventually enter and remain in D. Hence, the basin of attraction of the Lorenz attractor (19) contains a large region of R^3.

5.6 Absorbing Set and Bifurcation Analysis of the Lorenz System

We define the Lyapunov function

$$V(x, y, z) = \frac{1}{2}(x^2 + y^2 + (z - \rho - \sigma)^2). \quad (34)$$

Then, it is shown that $\frac{dV}{dt} \leq 0$. (35)

Therefore, all trajectories eventually enter a bounded ellipsoidal region, which serves as an absorbing set. The system possesses a strange attractor $\mathcal{A}$. Its basin of attraction consists of almost all initial conditions in R^3 except a set of measure zero associated with unstable invariant manifolds of equilibria. The basin boundary may exhibit complicated (possibly fractal) geometry near saddle equilibria. The parameter ρ is typically taken as the bifurcation parameter. The system has the equilibria:

$$E_0 = (0,0,0),$$

$$E_{\pm} = \left(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho - 1\right), \quad \rho > 1\,. \tag{36}$$

At the origin E0, the Jacobian matrix is:

$$J(E_0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}. \tag{37}$$

For $\rho < 1, all$ eigenvalues have negative real parts, so E0 is asymptotically stable. The basin of attraction exists corresponding to the stable equilibrium $E_0 = (0,0,0)$.

5.7 Pitchfork Bifurcation at ρ = 1

The characteristic equation is

$$\lambda^3 + (\sigma + \beta + 1)\lambda^2 + \beta(\sigma + \rho)\lambda + 2\beta\sigma(\rho - 1) = 0. \tag{38}$$

At $\rho = 1$, one eigenvalue crosses zero, and a supercritical pitchfork bifurcation occurs. Two new equilibria $E_{\pm}$ emerge for $\rho > 1$.

5.8 Hopf Bifurcation at $E_{\pm}$

For $\rho > 1$, the equilibria $E_{\pm}$exist. Linearization about $E_{\pm}$ leads to a cubic characteristic equation whose coefficients depend on σ, ρ, β. A Hopf bifurcation occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis. The critical value is

$$\rho_H = \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1}. \tag{39}$$

For the parameters, $\sigma = 10, \beta = \frac{8}{3}$, the Hopf bifurcation occurs at $\rho_H \approx 24.74.$ For $1 < \rho < \rho_H$ (before Hopf bifurcation), the basin splits between the two stable equilibria $E_{\pm}$. For $\rho > \rho_H$, the equilibria $E_{\pm}$ lose stability, and a strange attractor is formed. At this stage, the chaotic attractor emerges. At these parametric values, the Lorenz system shows a sequence of bifurcations leading the route to chaos from steady-state behavior to chaotic dynamics.

6 Conclusion

In this paper, we have presented the fundamental concepts of spectral analysis and numerical range in the setting of dynamical systems on Banach spaces. The notion of Li–Yorke chaos in Banach spaces have been redefined, and the associated definitions, theorems, and corollaries are reformulated and rigorously established. We developed spectral and numerical approaches to the study of chaotic dynamical systems using tools such as Lyapunov exponents, equilibrium analysis, and stability criteria. The relationship between spectral properties, numerical range, and the onset of chaos was examined, showing how instability and chaotic behavior arise naturally from the operator-theoretic structure of the system. Special attention was given to the Lorenz system formulated in a Banach space framework. Various equilibrium points have been analyzed, and their stability properties were investigated through spectral methods. The existence of chaos has been verified using basin of attraction analysis, Poincare sections, strange attractors, bifurcation analysis at different equilibrium points, and dissipative arguments ensuring the existence of bounded attractors. Thus, the results demonstrate that spectral analysis and numerical range theory provide powerful and systematic tools for understanding stability and chaos in infinite-dimensional dynamical systems. These methods offer a unified operator-theoretic framework for analyzing nonlinear phenomena arising in mathematical physics, fluid dynamics, and related applied fields. Extending Lorenz-type systems to Banach spaces provides a unified framework for studying stability, transient growth, and chaos in infinite-dimensional systems. Spectral theory determines asymptotic behavior, while numerical range analysis reveals transient instability mechanisms that drive chaotic dynamics beyond classical eigenvalue criteria.

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