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Construction of Not Distributive anti - Lattice Over a Permutation Group

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Abstract

The source of this discussion is the definition $f(ab) = f(b)f(a)$ by name anti – Lattice. It can be seen that there are *n*! permutations on a set of *n* symbols and also that some permutation fixes some symbols. Some permutations fix no symbols, some fix only one symbol, some fix 2 symbols, and continuing this, the identity permutation fixes all symbols. Basing on the fixation of symbols, a relation on the set of permutations defined as follows. Keeping the fixation of equal number of symbols as one class of permutations, the *n*! permutations in S_n are divided into equivalence classes that show reflexive, symmetry and transitive properties between equivalence classes. So, to construct a lattice with reference to this equivalence relation is termed as the anti – lattice while the join and meet operations are also not abelian. Further, this set also fails to hold the distributive property with reference to these join and meet operations and thus forming the Not Distributive anti – Lattice.

Keywords: anti – lattice, equivalence classes, cycles, product of transpositions, symmetric group of order *n*.

1. Introduction

A bijection from a set of *n* symbols onto itself is a permutation. There can be *n*! permutations on *n* symbol set and let this collection of permutations be S_n and using the composition of mappings denoted by **o** as an operation between any two permutations in S_n, it can be seen that $(S_n, \mathbf{o})$ is a non abelian group. A permutation can be written as a cycle when there is a mapping $i \to j \to k \to i$ within itself. This cycle can otherwise be written as (i, j, k). If (i, j) is the only cycle, then it is of length 2 and so, it is called a transposition. Every cycle is either a transposition or a product of transpositions. Depending on the equal number of transpositions in each permutation are considered as equivalent by the following definitions.

Symmetric Group on *n* Symbols:

Definition: If $S = \{1,2, \dots, \dots, n\}$, S_n is the collection of permutations on *S*, $f: S^2 \to S_n$ defined by $f(i,j) \to A_{i,j}$ where $A_{i,j}$ is a transposition. 1.1

Note that $f = (i, j, k) = (i, k)\circ(i, j)$ is the cycle written as the composition of transpositions that have *j* in common and so, the *f* is written as the composition of joint transpositions.

Cycles and Product of Transpositions:

The set of all permutations on *n* symbols is denoted by S_n

The ordering relation $\leq$ on S_n is defined by $(i, j) \leq (i, j, k)$ and a permutation formed by 0 transpositions is the minimal element in S_n 1.2

Unlike this, $f = (i, j)\circ(m, n)$ is a cycle that is written as the composition of two disjoint transpositions.

Ordering Relation on S_n

Definition: Let the relation ~ on S_n be $A_{i,j} \sim B_{i,j} \Leftrightarrow A_{i,j}\&B_{i,j}$ fixes the same indices $i\&j$ for any $A_{i,j}\&B_{i,j} \in S_n$. 1.3

For instance, A_3 and A_4 fixes the same indices in the adjoining table showing 5! = 120 permutations on 5 symbols.

Note that, there can be cycles that can be written as composition of transpositions which may be joint or disjoint.

2. Classifying the Permutations on 5 Symbols into Equivalence Classes:

Depending on the number of transpositions, the different classes due to partition are $P_0 = \{A_0\}$



Figure 2.1

$|P_0| = 1$ and P_0 stands for the permutation written as product of zero transpositions. 2.1

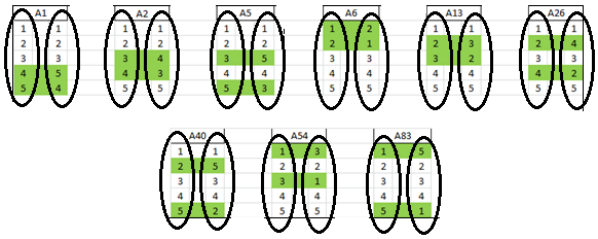


Figure 2.2

$P_1 = \{A_1, A_2, A_5, A_6, A_{13}, A_{26}, A_{40}, A_{54}, A_{83}\},$

$|P_1| = 9$

P_1stands for the set of permutations that are having only one transposition and the remaining part of the mapping is the identity mapping.

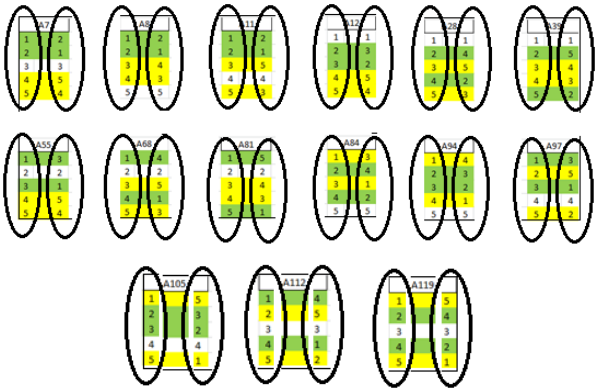
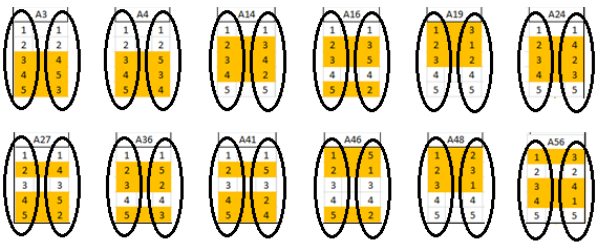


Figure 2.3

$P_{22} = \{A_7, A_8, A_{11}, A_{12}, A_{28}, A_{39}, A_{55}, A_{68}, A_{81}, A_{84}, A_{94}, A_{97}, A_{105}, A_{112}, A_{119}\}$

$|P_{22}| = 15$

Note that P_2 stands for the set of permutations that are written as two disjoint transpositions and one symbol following the identity mapping.



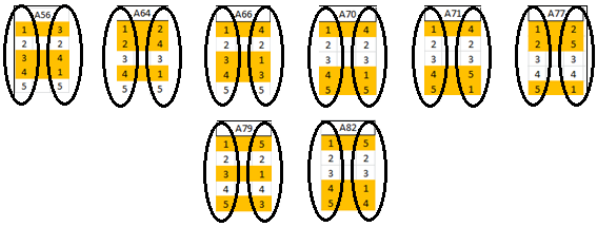


Figure 2.4

$P_3 = \{A_3, A_4, A_{14}, A_{16}, A_{19}, A_{24}, A_{27}, A_{36}, A_{41}, A_{46}, A_{48}, A_{56}, A_{58}, A_{64}, A_{66}, A_{70}, A_{71}, A_{77}, A_{79}, A_{82}\}$

$|P_3| = 20$

P_3 stands for the set of permutations that are having only cycles of length 3 which can be written as the composition of two transpositions that have joint.

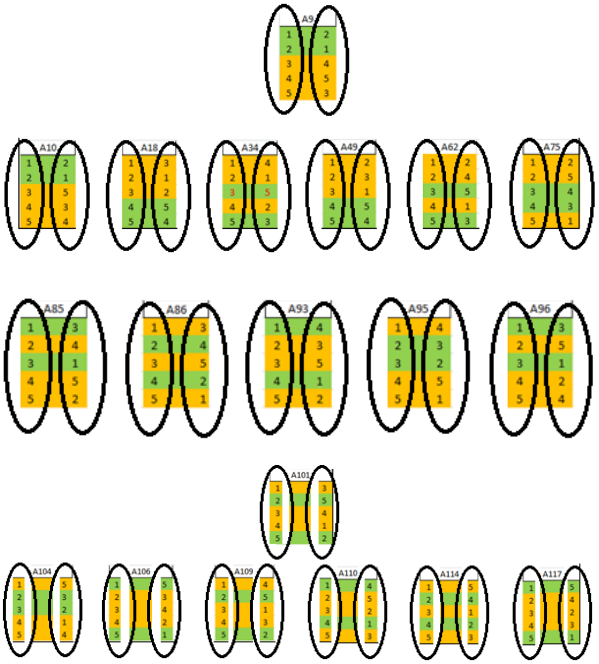
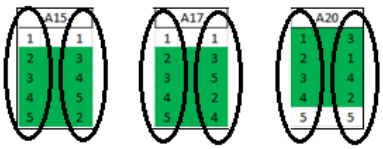


Figure 2.5

$P_{23} = \{A_9, A_{10}, A_{18}, A_{34}, A_{49}, A_{62}, A_{75}, A_{85}, A_{86}, A_{93}, A_{95}, A_{96}, A_{101}, A_{104}, A_{106}, A_{109}, A_{110}, A_{114}, A_{117}\}$

$|P_{23}| = 19$

P_{23} stands for the permutations having one transposition disjoint from one cycle of length 3 that can be written as the composition of two joint transpositions.



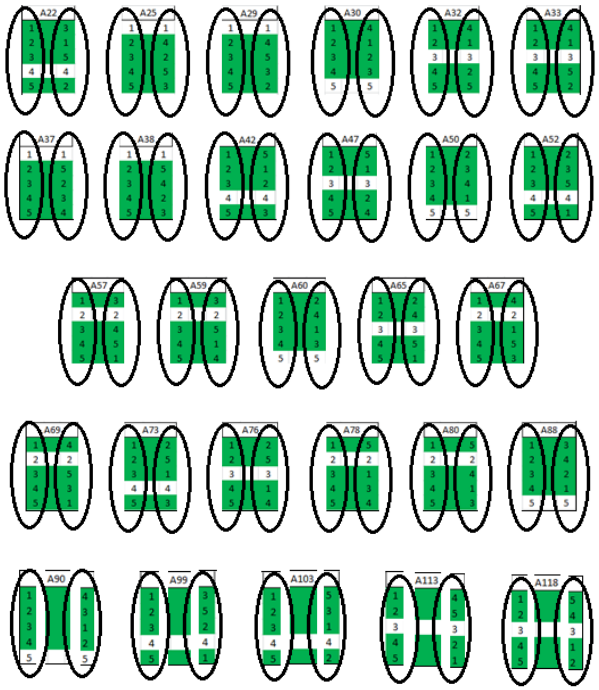


Figure 2.6

$P_4 = \{A_{15}, A_{17}, A_{20}, A_{22}, A_{25}, A_{29}, A_{30}, A_{32}, A_{33}, A_{37}, A_{38}, A_{42}, A_{47}, A_{50}, A_{52}, A_{57}, A_{59}, A_{60}, A_{65}, A_{67}, A_{69}, A_{73}, A_{76}, A_{78}, A_{80}, A_{88}, A_{90}, A_{99}, A_{103}, A_{113}, A_{118}\}$,

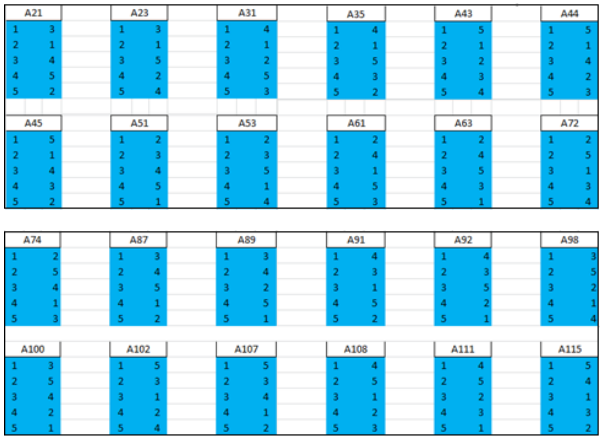
$|P_3| = 31$

P_4 stands for the set of cycles of length 4 that can be written as the composition of three joint transpositions.

$P_5 = \{A_{21}, A_{25}, A_{31}, A_{35}, A_{41}, A_{44}, A_{45}, A_{51}, A_{53}, A_{61}, A_{63}, A_{72}, A_{74}, A_{87}, A_{89}, A_{91}, A_{92}, A_{98}, A_{100}, A_{102}, A_{107}, A_{108}, A_{111}, A_{115}, A_{116}\}$

$|P_5| = 25$

P_5 stands for the set of cycles of length 5 that can be written as the composition of four joint transpositions.



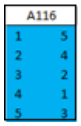


Figure 2.7

From the above permutation notations, it can be seen that $S_5 = \{A_0, \dots, A_{119}\}$

It may be followed that relation $\leq$ on S_5 based on number of transpositions has created a partition over S_5.

However, the relation is reflexive, symmetric and symmetric leading to an equivalence relation on S_5 and the equivalence classes consist of different permutations unlike the cosets of a subgroup or a quotient ring.

Definition: upon the set *A*, let the join operation $\vee$ be defined by $A_i \vee A_j = (A_i \circ A_j) \circ A_0$ where A_0 is the identity permutation shown in Fig.2.1. 2.2

Note that $A_i \circ A_j \neq A_j \circ A_i$ while the set of permutations on the set of *n* symbols is a non abelian group. However, the failure of $A_i \vee A_j = A_j \vee A_i$ creates an algebra that not even stands as a lattice. 2.3

Take $A_{41}$and $A_{79} \in P_3$.

$A_{41} \vee A_{79} =$

$A_{41} \circ A_{79} = A_{108} \in P_5$

= (14253)

= (1,4)∘ (1,2)∘ (1,5)∘(1,3)

On the other hand, $A_{79} \vee A_{41} =$

$A_{79} \circ A_{41} = A_{102} \in P_5$

$= (1,5,4,2,3)$

= (1,5)∘ (1,4)∘ (1,2)∘(1,3)

From this, it is clear that $A_{ijm} \vee A_{pqn} \neq A_{pqn} \vee A_{ijm}$

However, $A_{ijm} \vee A_{pqn}, A_{pqn} \vee A_{ijm} \in P_{ab}$

In particular, $\vee$ is abelian upto the equivalence class. 2.4

Definition: the meet operation $\wedge$ on *A* is defined by $A_i \wedge A_j = (A_i \circ A_j) \circ A_{119}$ 2.5

Same as the observation 2.4, in this definition also, $A_i \circ A_j \neq A_j \circ A_i$ forces

$A_i \wedge A_j \neq A_j \wedge A_i$. 2.6

However, as shown in (2.4), in the case of meet also, the abelian property holds upto equivalence class.

2.7

Definition: A Lattice $(\mathcal{L},\vee,\wedge)$ is said to be an anti – Lattice if the join and meet operations do not satisfy the abelian property and satisfy it upto the equivalence class 2.8

However, associative law and absorption laws are satisfied in routine.

3. Failure of Distributive Property:

The 0 cycles permutation is the least and the 4 cycle permutation is the largest will show the ordering of the members of the set S_n will satisfy the other conditions of a lattice. So,

Take $A_{98} \in P_5$ and $A_{14} \in P_3$.

$A_{98} \circ A_{14} = A_{55} \in P_{22}$

$A_{55} \circ A_{119}$=

So, $A_{98} \wedge A_{14} = A_{108}$

Consider $(A_{54} \vee A_{75}) \wedge A_{56}$

$= (A_{74}) \wedge A_{56}$

$= (A_{74} \circ A_{56}) \circ A_{119}$

$= A_{55}$

= (3, 1)∘ (5, 4) 3.1

Consider $(A_{54} \wedge A_{56}) \vee (A_{75} \wedge A_{56})$

$= \{(A_{54} \circ A_{56}) \circ A_{119}\} \vee \{(A_{75} \circ A_{56}) \circ A_{119}\}$

$= (A_2 \circ A_{119}) \vee (A_{113} \circ A_{119})$

$= A_{106} \circ A_1$

$= A_{107}$

=(5,3)∘ (5,4)∘ (5,1)∘ (5,2) 3.2

Observe from (3.1) and (3.2) that

$(A_{54} \vee A_{75}) \wedge A_{56} \neq (A_{54} \wedge A_{56}) \vee (A_{75} \wedge A_{56})$

Therefore, *L* is not a distributive lattice.

Even though, the constructed algebra $\mathcal{L}=S_5$ is an anti – Lattice, the distributive property failure is shown. Also, the ordering relation portioning this anti – lattice appears to be a symmetric relation. But, it's an anti – symmetric relation that satisfies the basic partially ordered set structure to construct the lattice. Thus, the constructed lattice satisfies only the closure properties of join and meet along, right from absorption laws, all other parts of the definition of Lattice are a failure in this algebra forming an anti – lattice.

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